§21 Maps of Spheres

Recall $B^{n+1} = \{(x_1, \ldots, x_{n+1}) \mid \sum x_i^2 \le 1\}$
$S^n = \{(x_1, \ldots, x_{n+1}) \mid \sum x_i^2 = 1\}$.

We define their homology gps to be those of any simp. comp. with homeomorphic underlying spaces. From Section 8, B^{n+1} is acyclic and $H_n(S^n) \cong \mathbb{Z}$.

Def Let $n \ge 1$. Let $f: S^n \to S^n$ be cont. Let α be a generator of $H_n(S^n)$. Then $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$. We define the degree of f to be d and write $\deg f = d$.

Facts
① If $f \simeq g$ (homotopic), then $\deg f = \deg g$.
② If f extends to a cont. map $h: B^{n+1} \to S^n$, then $\deg f = 0$.
③ The degree of the identity map is 1.
④ $\deg f \circ g = (\deg f) \cdot (\deg g)$.

Note In the complex plane the map $z \to z^d$ wraps S^1 around itself d times.

① By Thm 19.2, $f_* = g_*$. Thus $\deg f = \deg g$.
② Suppose $h: B^{n+1} \to S^1$ is cont. and $h|\partial B^{n+1} = f$.
Let $j: S^n \to \partial B^{n+1} \subset B^{n+1}$ be inclusion. Now $f = h \circ j$.
Thus, $f_* = h_* \circ j_*$. Thus $f_*(\alpha) = h_*(j_*(\alpha)) = 0$ since $H_n(B^{n+1}) = 0$.
③ & ④ are clear from Thm 18.1 (functor properties.)

Thm 21.1 There is no retraction $r: B^{n+1} \to S^n$.

Pf Suppose $r: B^{n+1} \to S^n$ is a retraction.
Then $r|\partial B^{n+1} \to S^n$ is the identity.
Hence r is a cont. extention of the $id: S^n \to S^n$.
But then (2) $\Rightarrow$ deg id $= 0$ contradicting (3). $\square$

Note See Thm 55.2 in Munkres "Topology"; Also Exercises 9 and 10 in §58.

Thm 21.2 (Brouwer fixed-point thm) Every cont. map $\phi: B^n \to B^n$ has a fixed point.

Pf Suppose $\phi: B^n \to B^n$ has no fixed pt. Define $h: B^n \to S^{n-1}$ by

$$\overset{\text{vector}}{h(x)} = \frac{x - \phi(x)}{\underbrace{\|x - \phi(x)\|}_{\text{never zero}}}.$$

Let $f: S^{n-1} \to S^{n-1}$ be $h|\partial B^n$.
Then $\deg f = 0$.

We show f is homotopic to the id.
Let $H: S^{n-1} \times I \to S^{n-1}$ be

$$H(x,t) = \frac{x - t\phi(x)}{\|x - t\phi(x)\|}.$$

Then $H(x,0) = \frac{x}{\|x\|} = x$ on S^{n-1} and $H(x,1) = h(x)$.

But we need to check that $\|x - t\phi(x)\|$ is never zero.
The $t=1$ we already have this. Let $0 \le t < 1$.
We know $\|x\| = 1$ and $\|\phi(x)\| \le 1$ ~~since these are points on S^{n-1}.~~ ~~But now $\|t\phi(x)\|$~~
Since $x \in S^{n-1}$ and $\phi(x) \in B^n$.
Then $\|t\,\phi(x)\| = t\,\|\phi(x)\| \le t < 1$.

Thus $x - t\phi(x) \neq 0$.

Since $f \simeq id$, $\deg f = 1$. Contradiction. $\square$

<u>Def</u> The <u>antipodal map</u> $a: S^n \to S^n$ is $a(x) = -x$.

<u>Thm 21.3</u> $\deg a = (-1)^{n+1}$.

<u>Outline of Pf</u> $a(x_1, \ldots, x_{n+1}) = (-x_1, -x_2, \ldots, -x_{n+1})$.
Let $\rho_i(x_1, \ldots x_{n+1}) = (x_1, \ldots, -x_i, \ldots x_{n+1})$.

Show $\deg \rho_i = -1$. Then $\deg a = (-1)^{n+1}$.

Let K be any poly. model of the ith equator of S^n, $S^{n-1} = \{(x_1, \ldots, 0, \ldots x_{n+1}) \mid (x_1, \ldots x_n) \in S^n\}$.
Let $w_0 = (0, 0, \ldots, 1, 0 \ldots 0)$ and $w_1 = (0, \ldots 0, -1, 0, \ldots 0)$.
Then $L = (w_0 * K) \cup (w_1 * K)$ is a poly model for S^n.
Define a map $r: |L| \to |L|$ that fixes $|K|$ and exchanges w_0 and w_1 and extends linearly to $|L|$.

Claim $\deg r = -1$. (The construction in the book show that $\deg r = \deg p_i$.)

Let z be an n-cycle in L. Then z can be written as

$$z = [w_0, c_m] + [w_1, d_m]$$

where c_m, d_m are $n-1$-chains in K. Assume $n>1$. Since $\partial z = 0$ and $\partial z = c_m - [w_0, \partial c_m] + d_m - [w_1, \partial d_m]$ we have

$$c_m + d_m - ([w_0, \partial c_m] + [w_1, \partial d_m]) = 0.$$

Now $c_m + d_m$ is carried by K and $[w_0, \partial c_m] + [w_1, \partial d_m]$ has no $n-1$ simplices in K. Thus $c_m + d_m = 0$, or $c_m = -d_m$. Now

$$z = [w_0, c_m] - [w_1, c_m].$$

But $r_\#(z) = [w_1, c_m] - [w_0, c_m] = -z$.

(This is also true in the case $n=1$, which you can check.)

Thus $r_*(\alpha) = -\alpha$. Thus, $\deg r = -1$. $\boxtimes$

Thm 21.4 If $h: S^n \to S^n$ has degree $\neq (-1)^{n+1}$, then h has a fixed point.

Pf Suppose $h: S^n \to S^n$ has no fixed point.
Let $H: S^n \times I \to S^n$ be

$$H(x,t) = \frac{(1-t)h(x) + t(-x)}{\|(1-t)h(x) + t(-x)\|}.$$

Then $H(x,0) = \dfrac{h(x)}{\|h(x)\|} = h(x)$ and

$$H(x,1) = \frac{-x}{\|-x\|} = -x = a(x).$$

We need to show $\|(1-t)h(x) + t(-x)\|$ is never zero.

Suppose $(1-t)h(x) = tx$.

$\Rightarrow$ $(1-t)\|h(x)\| = t\,\|x\|$

$\Rightarrow$ $1-t = t$

$\Rightarrow$ $t = \frac{1}{2}$

$\Rightarrow$ $\frac{1}{2}h(x) = \frac{1}{2}x$

$\Rightarrow$ $h(x) = x$,

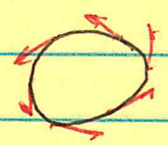
which never happens! $\square$

Thm 21.5 Let $h: S^n \to S^n$ have degree ± 1.
Then $\exists\, x \in S^n$ s.t. $h(x) = -x$.

Pf: Let $f = a \circ h$. $\deg f = (-1)^{n+1} \cdot \deg h \neq (-1)^{n+1}$. Thus, by Thm 21.4, f has a fixed pt x; $f(x) = x$. Then

$$a(h(x)) = x \quad \text{and} \quad a(h(x)) = -h(x) \Rightarrow h(x) = -x. \quad \boxed{\circ}$$

Cor 21.6 (The Hairy Ball Thm) S^n has a non-zero (never-zero) tangent vector field iff n is odd.

Pf For $n=1$ use . For $n = 2k-1$ let

$$v(x_1, x_2, \dots x_{2k-1}, x_{2k}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{2k}, x_{2k-1}).$$

Then $\|v(x)\| = 1$ and $v(x) \cdot x = 0 \ \forall x \in S^n$. Thus $v(x)$ is tangent everywhere and zero nowhere on S^n.

Now, suppose S^n has a non-zero tangent v.f., $v(x)$.
Let $h: S^n \to S^n$ be $h(x) = v(x) / \|v(x)\|$.

$\forall x \in S^n$, $v(x) \cdot x = 0$. Thus $v(x) \neq x$. Thus $h(x) \neq x$, $\forall x \in S^n$.
By Thm 21.4 $\deg h = (-1)^{n+1}$.

$\forall x \in S^n$ $v(x) \neq -x$ (since $v(x) \cdot x = 0$). Thus $h(x) \neq -x$ $\forall x \in S^n$.
By Thm 21.5 $\deg h = 1$.

Thus $1 = (-1)^{n+1} \Rightarrow n$ is odd.

You know a thm is important when it is on youtube.