§ 22 The Hopf Trace Thm and The Lefschetz Fixed-Point Thm

Def Let $A = [a_{ij}]^{n\times n}$. $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

Fact $\operatorname{tr} AB = \operatorname{tr} BA \Rightarrow \operatorname{tr} PAP^{-1} = \operatorname{tr} A$.
Thus, $\operatorname{tr} A$ is an invariant under change of bases.

Def Let G be a finitely generated free abelian group. Let $\phi: G \to G$ be a hom. Let M_ϕ be a matrix representing ϕ w.r.t. some basis. Define $\operatorname{tr}\phi = \operatorname{tr} M_\phi$.

Thm 22.1 (The Hopf Trace Thm) Let K be a finite complex. Let $\phi: C(K) \to C(K)$ be a "self" chain map. Then

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr} \phi_p : C_p(K) \circlearrowleft \;=\; \sum_{p=0}^{n} (-1)^p \operatorname{tr}\left(\phi_{p*} : \frac{H_p}{T_p} \circlearrowleft\right).$$

Note You know each $\phi_p: C_p(K) \to C_p(K)$ induces a hom. $\phi_{p*}: H_p(K) \to H_p(K)$. This map induces a hom $\phi_{p\#}: \frac{H_p}{T_p} \to \frac{H_p}{T_p}$. You can think of it as the restriction of ϕ_{*p} to the torsion free part of $H_p(K)$.

Def The proof will use the group $W_p(K)$ of <u>weak p-boundaries</u>; $W_p(K) = \{c_p \in C_p(K) \mid \exists\, d_{p+1} \in C_{p+1}(K),\ m \in \mathbb{N}$ s.t. $\partial d_{p+1} = m c_p\}$.

Clearly $B_p(K) \subset W_p(K)$. But also $W_p(K) \subset Z_p(K)$ since $\partial m c_p = 0 \Rightarrow \partial c_p = 0$.

Application of the HTT (Thm 22.2)

Suppose $\phi = id$. Then $tr(id : C_p(K) \circlearrowleft) = \operatorname{rank} C_p(K)$.

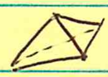
Let $\chi(K) = \sum_{p=0}^{n} (-1)^p \operatorname{rank}(C_p(K)) = \sum_{p=0}^{n} (-1)^p \beta_p$,

where $\beta_p = \operatorname{rank} H_p(K)/T_p(K)$; it is called the p-th betti number.

$\chi(K)$ is the Euler number or characteristic of K.

If $|K| \overset{\text{homotopy eq}}{\simeq} |L|$, then $\chi(K) = \chi(L)$. Thus χ is a homotopy invariant.

Ex

$T =$  $\chi(T) = 4 - 6 + 4 = 2$.

$C =$  $\chi(C) = 8 - 12 + 6 = 2$.

See Exercise 5.

Outline of Proof of Hopf Trace Thm

Step 1 Let G be a free abelian group with finite rank. Let H be a subgp and suppose G/H is free. Let $\phi : G \to G$ be hom. that carries H into H. Then ϕ induces hom's $G/H \to G/H$ and $H \to H$ that we also denote by ϕ. Then

$$\mathrm{tr}\,(\phi : G \circlearrowright) = \mathrm{tr}\,(\phi : \tfrac{G}{H} \circlearrowright) + \mathrm{tr}\,(\phi : H \circlearrowright).$$

Pf: Algebra.

Step 2 We want to apply step 1 to $\phi_p : C_p \to C_p$ with

$$B_p \subset W_p \subset Z_p \subset C_p.$$

Since ϕ_p commutes with ∂ it takes each subgp to itself.

Step 3 C_p/Z_p is free and $\mathrm{tr}(\phi : C_p/Z_p \circlearrowright) = \mathrm{tr}(\phi : B_{p-1} \circlearrowright)$

Step 4 Z_p/W_p is free and $\mathrm{tr}(\phi ; Z_p/W_p \circlearrowright) = \mathrm{tr}(\phi : H_p/T_p \circlearrowright)$.

Step 5 $\mathrm{tr}(\phi : W_p \circlearrowright) = \mathrm{tr}(\phi : B_p \circlearrowright)$.

Back to Step 2

$$tr(\phi : C_p \mathbb{Q}) = tr(\phi : C_p/Z_p^{\mathbb{Q}}) + tr(\phi : Z_p^{\mathbb{Q}})$$

$$= tr(\phi ; B_{p-1}) + tr(\phi : Z_p/W_p) + tr(\phi : W_p)$$

$$= tr(\phi : B_{p-1}) + tr(\phi : H_p/T_p) + tr(\phi, B_p).$$

Step 6 $\sum_{p=0}^{n} (-1)^p \, tr(\phi : C_p) =$

$$\underbrace{tr(\phi : B_{-1})}_{=0} + \sum_{p=0}^{n} (-1)^p \, tr(\phi ; H_p/T_p) + \underbrace{tr(\phi, B_n)}_{=0}$$

That is the result we wanted.

Details of Step 3: Claim: $C_p/Z_p \cong B_{p-1}$

Pf $\partial_p : C_p \to C_{p-1}$. By def. im $\partial_p = B_{p-1}$. So

$\partial_p : C_p \to B_{p-1}$ is well defined and onto.

Ker $\partial_p : C_p \to B_{p-1}$ is still Z_p.

Thus, ∂_p induces and iso. $\partial_p : \frac{C_p}{Z_p} \to B_{p-1}$.

Thus C_p/Z_p is free since B_{p-1} is free.

Consider the commutative diagram.

$$\begin{tikzcd}
C_p/Z_p \arrow[r, "\partial_p"', "\cong"] \arrow[d, "\phi_p"'] & B_{p-1} \arrow[d, "\phi_{p-1}"] \\
C_p/Z_p \arrow[r, "\partial_p", "\cong"'] & B_{p-1}
\end{tikzcd}$$

$$\mathrm{tr}\left(\phi_p : \tfrac{C_p}{Z_p}\right) = \mathrm{tr}\left(\underset{\text{bottom}}{\partial_p} \circ \phi_p \circ \underset{\text{top}}{\partial_p^{-1}} : B_{p-1} \circlearrowleft\right) = \mathrm{tr}\left(\phi_{p-1} : B_{p-1}\right)$$

Read Steps 4 and 5 on your own. 4 is similar to 3. 5 is a bit harder. $\square$

Def Let K be a finite complex, let $h : |K| \to |K|$ be continuous. Then

$$\Lambda(h) = \sum_{p=0}^{n} (-1)^p \operatorname{tr}\left(h_* : H_p(K)/T_p(K) \circlearrowleft\right)$$

is called the **Lefschetz number** of h.

Note This generalize $\chi(K) = \Lambda(id)$. $\Lambda(h) = \Lambda(h')$ if $h \simeq h'$ and ~~is a top. invariant~~: does depend on K, only $|K|$.

Thm 22.3 (Lefschetz Fixed-Point Thm).

Let K be a finite complex, let $h : |K| \to |K|$ be cont. If $\Lambda(h) \neq 0$, then h has a fixed point.

Outline **Step 1** $\exists$ a subd. K' of K s.t.

$$h(\overline{St(v, K')}) \cap \overline{St(v, K')} = \phi \quad \forall v \in K'^{(0)}.$$

Step 3 Let K'' be a subd. of K' s.t. h has a simp. approx. $f : K'' \to K'$

Suppose τ is a simplex of K'' and σ is a simplex of K' with $\tau \subset \sigma$. Then Step 1 $\Rightarrow f(\tau) \neq \sigma$.

Step 3 Let $\lambda : \mathcal{C}(K') \to \mathcal{C}(K'')$ be the subdivision operator. (see pg 96) Now our simp. approx. $f : K'' \to K'$ induces a hom. on the chain gps, $f_\# : \mathcal{C}(K'') \to \mathcal{C}(K')$. Let $\phi : \mathcal{C}(K') \to \mathcal{C}(K')$ be $\phi = f_\# \circ \lambda$. Then ϕ induces a hom $\phi_* : H_p(K') \to H_p(K')$ which is h_* by definition (pg 160). ~~And~~

$$\Lambda(h) = \sum_{p=0}^{n} (-1)^p \, tr\left(h_* : \frac{H_p(K')}{T_p(K')} \right) = \sum_{p=0}^{n} (-1)^p \, tr(\phi : C_p(K) \circlearrowleft)$$

But $tr(\phi : C_p(K) \circlearrowleft) = 0$ since the matrix M_ϕ, using the standard basis of p-simplices, has only zeros on its main diagonal. $\Box$

Lem 22.4 Let K be a finite ~~comple~~ complex with $|K|$ connected. Let $h: |K| \to |K|$ be cont. Then $h_*: H_0(K) \to H_0(K)$ is the identity.

Pf See textbook

Thm 22.5 Let K be a finite complex. Let $h: |K| \to |K|$ be cont. If $|K|$ is acyclic, then h has a fixed pt.

Pf $H_0(K) \cong \mathbb{Z}$. h_* is the identity. Thus

$$\operatorname{tr}(h_*: H_0(K) \circlearrowleft) = 1.$$

Since all the higher dim. homology gps are trivial $\Lambda(h) = 1$. By the L.F.P.T. h has a fixed pt. $\square$

Thm 22.6 The deg $(a) = (-1)^{n+1}$.

Pf See textbook.

Exercise #1 Show every cont. $f: P^2 \to P^2$ has a fixed pt.

Pf $H_0(P^2) \cong \mathbb{Z}$, $H_1(P^2) \cong \mathbb{Z}/2\mathbb{Z}$, $H_2(P^2) = 0$ (pg 38).

By Lemma 22.4 $f_*: H_0 \to H_0$ is the identity. Thus $\operatorname{tr}(f_*: H_0 \circlearrowleft) = 1$. Since $H_1/T_1 = 0$ and $H_2 = 0$

$$\Lambda(f) = 1 - 0 + 0 = 1 \neq 0.$$

Thus by the LFPT f has a fixed pt. $\square$