§26 The Eilenberg - Steenrod Axioms.

Def An admissible class of pairs is a collection $\mathcal{A}$ of pairs of top. spaces (X, A), where $A \subset X$, s.t.

① If $(X,A) \in \mathcal{A}$ so are (X,X), $(X,\emptyset)$, (A,A) and $(A,\emptyset)$.
Note: $(A,\emptyset)$ is usually denoted A.

② If $(X,A) \in \mathcal{A}$ so is $(X \times I, A \times I)$ [See lemma 19.1]

③ $\exists$ a one-point space P s.t. $(P,\emptyset) \in \mathcal{A}$.

Def A homology theory for an admissible class $\mathcal{A}$ consists of 3 families of functions indexed by $\mathbb{Z}$ that satisfy 8 axioms.

The functions are

① $\forall p \in \mathbb{Z}$, $H_p : \mathcal{A} \to \{\text{abelian gps}\}$, $(X,A) \to H_p(X,A)$.

② $\forall p \in \mathbb{Z}$ $*_p$: (cont. functions between pairs in $\mathcal{A}$) $\to$
(homomorphisms between their homology gps.)

$$*_p(f : (X,A) \to (Y,B)) = f_{*_p} : H_p(X,A) \to H_p(Y,B).$$

③ $\forall p \in \mathbb{Z}$ $(X,A) \to \partial_{*p} : H_p(X,A) \to H_{p-1}(A,\emptyset)$

↑ connecting homomorphism

The axioms are

Functorial prop's
① $*(id) = id_*$ (id maps go to gp isomorphisms)
② $(k \circ h)_* = k_* \circ h_*$

③ If $f:(X,A) \to (Y,B)$ is cont. then

$$\begin{tikzcd} H_p(X,A) \arrow[r, "f_*"] \arrow[d, "\partial_*"] & H_p(Y,B) \arrow[d, "\partial_*"] \\ H_p(A) \arrow[r, "(f|A)_*"'] & H_{p-1}(B) \end{tikzcd}$$

, commutes.

④ The sequence: $\to H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X,A) \xrightarrow{\partial_*} H_{p-1}(A) \to$ where $i: A \to X$ and $\pi: (X,\phi) \to (X,A)$ are inclusions, is exact.

⑤ $h \simeq f \Rightarrow h_* = f_*$

⑥ [Excision] For $(X,A) \in \mathcal{Q}$, let $U \subset X$ be open s.t. $\bar{U} \subset A$. If $(X-U, A-U) \in \mathcal{Q}$, then the inclusion map

$$(X-U, A-U) \longrightarrow (X,A)$$

induces an iso $H_p(X-U, A-U) \cong H_p(X,A)$, $\forall p$.

⑦ If P is a one-point set $H_p(P) = 0$ for $p \neq 0$ and $H_0(P) \cong \mathbb{Z}$ (or G if using G coeff's.)

(8) (Axiom of compact support) $\forall\ \alpha \in H_p(X,A)$,

$\exists\ (X_0, A_0) \in \mathcal{a}$ s.t. $X_0 \subset X$, $A_0 \subset A$, X_0 and A_0 are compact s.t. α is in the image of

$$H_p(X_0, A_0) \to H_p(X,A)$$

induced by inclusion. (Every chain lives on a compact set.)

<u>Fact</u> Our simplicial homology does not quite fit this scheme. A simplicial complex is not a top. sp. A little work is needed. We use the underlying spaces, $|K|$, instead. Then §27 shows the axioms work. Read on your own.

<u>Fact</u> Many Thm's in homology depend only on the axioms. Exercise #3 shows the M-V seq follows from the axioms. Exercise #2 is the long exact seq. of a triple. #1 is the exact braid.

I will do #2 in class. It uses #1. which is in a handout from Bredon's book.