Braid Lemma

$$\begin{tikzcd}
 & A \arrow[rr, bend left] \arrow[dr] & & B \arrow[rr, bend left] \arrow[dr] & & C \arrow[rr, bend left] \arrow[dr] & & D \\
E \arrow[ur] \arrow[dr] & & F \arrow[ur] \arrow[dr] & & G \arrow[ur] \arrow[dr] & & H \arrow[ur] & \\
 & I \arrow[ur] \arrow[rr, bend right] & & J \arrow[ur] \arrow[rr, bend right] & & K \arrow[ur] & &
\end{tikzcd}$$

If $E \to A \to B \to G \to K$,

$E \to I \to J \to G \to C \to D$,

$A \to F \to J \to K \to H \to D$,

are exact, and

$I \to F \to B$ is zero,

then

$I \to F \to B \to C \to H$ is exact.

We call this a <u>exact braid</u>.

§26

#2 Let (X, A), (A, B) and (X, B) be admissible pairs in a homology theory. Let

$$\pi : (A, B) \longrightarrow (X, B) \; , \; \eta : (X, B) \hookrightarrow (X, A) \; , \; i : (A, \phi) \to (A, B)$$

be inclusions. Define a map $\beta : H_p(X, A) \to H_{p-1}(A, B)$ to be the composite, $\beta = i_* \circ \partial_*$,

$$H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(A, B)$$

where ∂_* is from Axiom 4. Then we can prove that

$$\cdots \xrightarrow{\beta} H_p(A, B) \xrightarrow{\pi} H_p(X, B) \xrightarrow{\eta} H_p(X, A) \xrightarrow{\beta} H_{p-1}(A, B) \xrightarrow{\pi} \cdots$$

is exact. (I dropped the $*$'s on π and η.)

pf A hint says to show that $H_p(A, A) = 0$. Using Axiom 4

$$H_p(A) \underset{\cong}{\xrightarrow{i_*}} H_p(A) \xrightarrow{\pi_*} H_p(A, A) \xrightarrow{\partial_*} H_{p-1}(A) \underset{\cong}{\xrightarrow{i_*}} H_{p-1}(A)$$

is exact. But by Axiom 1, both i_*'s are isomorphisms. Thus, $\operatorname{im} i_* = H_p(A) = \ker \pi_* \Rightarrow \operatorname{im} \pi_* = 0$. Also, $\operatorname{im} \partial_* = 0$. Thus, $H_p(A, A) = 0$.

From this it follows that $\eta \circ \pi$ is trivial (zero) since its image is in the subgp $H_p(A, A)$ of $H_p(X, A)$.

Consider the braid diagram.

$$\begin{tikzcd}[column sep=small]
& H_p(X) \arrow[rr, bend left] \arrow[dr] & & H_p(X,A) \arrow[rr] \arrow[dr] & & H_{p-1}(A,B) \arrow[rr] \arrow[dr] & & H_{p-2}(B) \\
H_p(A) \arrow[ur] \arrow[dr] & & H_p(X,B) \arrow[ur] \arrow[dr] & & H_{p-1}(A) \arrow[ur] \arrow[dr] & & H_{p-1}(X,B) \arrow[ur] & \\
& H_p(A,B) \arrow[ur] \arrow[rr, bend right] & & H_{p-1}(B) \arrow[ur] \arrow[rr, bend right] & & H_{p-1}(X) \arrow[ur] & &
\end{tikzcd}$$

By Axiom 4 the following sequences are exact.

$$H_p(A) \to H_p(X) \to H_p(X,A) \to H_{p-1}(A) \to H_{p-1}(X)$$

$$H_p(A) \to H_p(A,B) \to H_{p-1}(B) \to H_{p-1}(A) \to H_{p-1}(A,B) \to H_{p-2}(B)$$

$$H_p(X) \to H_p(X,B) \to H_{p-1}(B) \to H_{p-1}(X) \to H_{p-1}(X,B) \to H_{p-2}(B).$$

By our initial remarks

$$H_p(A,B) \to H_p(X,B) \to H_p(X,A)$$ is zero (trivial).

By the Braid Lemma

$$H_p(A,B) \to H_p(X,B) \to H_p(X,A) \to H_{p-1}(A,B) \to H_{p-1}(X,B)$$

is exact.