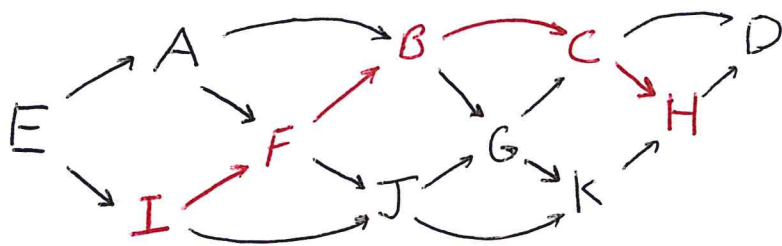


Braid Lemma



If $E \rightarrow A \rightarrow B \rightarrow G \rightarrow K$,
 $E \rightarrow I \rightarrow J \rightarrow G \rightarrow C \rightarrow D$,
 $A \rightarrow F \rightarrow J \rightarrow K \rightarrow H \rightarrow D$,
are exact, and

$I \rightarrow F \rightarrow B$ is zero,

then

$I \rightarrow F \rightarrow B \rightarrow C \rightarrow H$ is exact.

We call this a exact braid.

§26

#2 Let (X, A) , (A, B) and (X, B) be admissible pairs in a homology theory. Let

$$\pi: (A, B) \rightarrow (X, B), \quad \eta: (X, B) \rightarrow (X, A), \quad i: (A, \emptyset) \rightarrow (A, B)$$

be inclusions. Define a map $\beta: H_p(X, A) \rightarrow H_{p-1}(A, B)$ to be the composite, $\beta = i_* \circ \partial_X$,

$$H_p(X, A) \xrightarrow{\partial_X} H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(A, B)$$

where ∂_X is from Axiom 4. Then we can prove that

$$\cdots \xrightarrow{\beta} H_p(A, B) \xrightarrow{\pi_*} H_p(X, B) \xrightarrow{\eta_*} H_p(X, A) \xrightarrow{\beta} H_{p-1}(A, B) \xrightarrow{\pi_*} \cdots$$

is exact. (I dropped the $*$'s on π and η .)

pf A hint says to show that $H_p(A, A) = 0$. Using Axiom 4

$$H_p(A) \xrightarrow{i_*} H_p(A) \xrightarrow{\pi_*} H_p(A, A) \xrightarrow{\partial_X} H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(A)$$

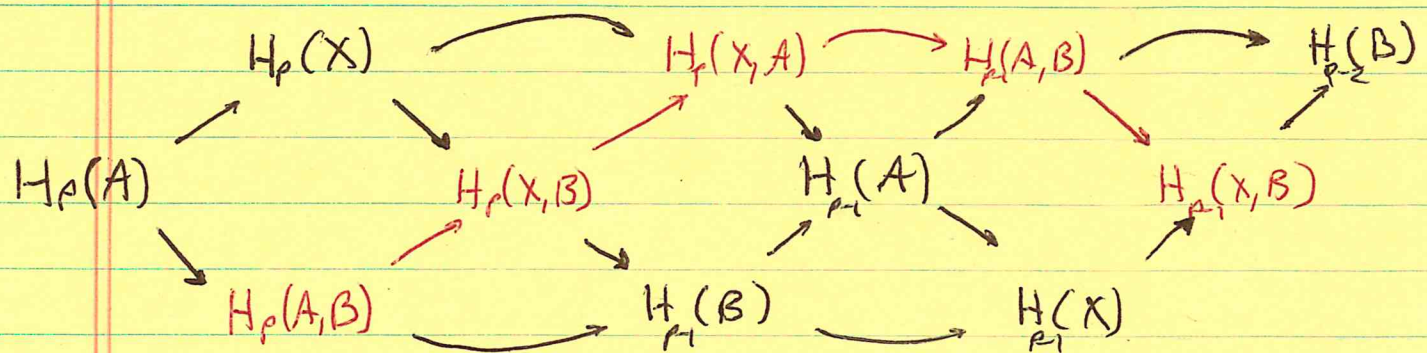
is exact. But by Axiom 1, both i_* 's are isomorphisms.

Thus, $\text{im } i_* = H_p(A) = \ker \pi_* \Rightarrow \text{im } \pi_* = 0$.

Also, $\text{im } \partial_X = 0$. Thus, $H_p(A, A) = 0$.

From this it follows that $\eta \circ \pi$ is trivial (zero) since its image is in the subgp $H_p(A, A)$ of $H_p(X, A)$.

Consider the braid diagram.



By Axiom 4 the following sequences are exact.

$$H_p(A) \rightarrow H_p(X) \rightarrow H_p(X, A) \rightarrow H_{p-1}(A) \rightarrow H_{p-1}(X)$$

$$H_p(A) \rightarrow H_p(A, B) \rightarrow H_{p-1}(B) \rightarrow H_{p-1}(A) \rightarrow H_{p-1}(A, B) \rightarrow H_{p-2}(B)$$

$$H_p(X) \rightarrow H_p(X, B) \rightarrow H_{p-1}(B) \rightarrow H_{p-1}(X) \rightarrow H_{p-1}(X, B) \rightarrow H_{p-2}(B).$$

By our initial remarks

$$H_p(A, B) \rightarrow H_p(X, B) \rightarrow H_p(X, A) \text{ is zero (trivial).}$$

By the Braid Lemma

$$H_p(A, B) \rightarrow H_p(X, B) \rightarrow H_p(X, A) \rightarrow H_{p-1}(A, B) \rightarrow H_{p-1}(X, B)$$

is exact.