§29 Singular Homology Groups

Rough Idea: Let X be any top. sp. Let Δ_p be the standard p-simplex in $\mathbb{R}^\infty$. For each p let

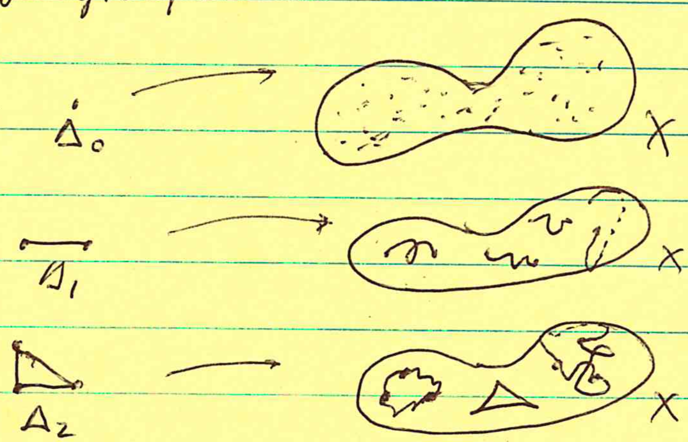
$S_p(X)$ = the free group generated by the collection of continuous function $\Delta_p \longrightarrow X$.

$S_p(X)$ is called the p-th singular chain group.
This is a huge group!

$S_0(X)$

$S_1(X)$

$S_2(X)$



We shall define a boundary map ∂, show that $\partial\partial = 0$, and then define homology groups.

Let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, etc.
and $e_0 = (0, 0, 0, \ldots)$.

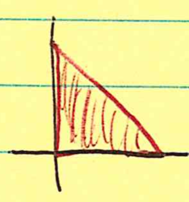
Let Δ_p be the p-simplex in $\mathbb{R}^\infty$ having vertices $e_0, e_1, e_2, \ldots, e_p$.

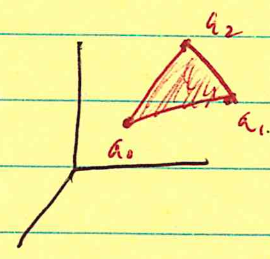
Def A <u>singular p-simplex</u> of a top sp. X is a cont. map $T: \Delta_p \to X$.
The <u>singular p-chain group</u> of X is the free abelian group generated by the set of all sing. p-simplices. It is denoted $S_p(X)$.

Some machinary is needed to define the boundary maps.

For any $p+1$ points, $a_0, a_1, \ldots, a_p$ in $\mathbb{R}^\infty$, $\exists$ a unique affine map $l: \Delta_p \to \mathbb{R}^\infty$ s.t. $l(e_i) = a_i$, $i = 0, 1, \ldots, p$.
It is given by

$$l(x_1, x_2, x_3, \ldots, x_p, 0, 0, \ldots) = a_0 + \sum_{i=1}^{p} x_i(a_i - a_0).$$

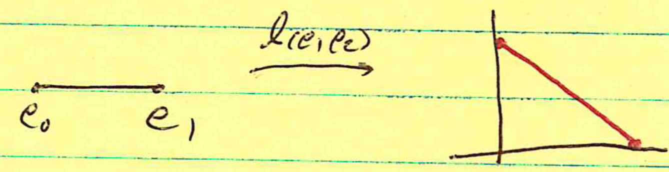




If it is clear what $a_0, a_1, \ldots, a_p$ are we use l to denote this function. If we need to be explicit we will use

$$l_{(a_0, a_1, \ldots a_p)} : \overset{\Delta_p}{(x_1, x_2, \ldots x_p, 0, 0, \ldots)} \longrightarrow \mathbb{R}^\infty.$$

Thus, $l(e_0, e_1, \ldots e_p)$ is the inclusion map $\Delta_p \hookrightarrow \mathbb{R}^\infty$.
Then $l(e_0, e_1, \ldots, \hat{e}_i, \ldots e_p)$ is the map taking Δ_{p-1} to the face of Δ_p opposite e_i.



Now if $T: \Delta_p \to X$ is a p-simplex, we can form the composition $T \circ l(e_0, \ldots \hat{e}_i \ldots e_p) : \Delta_{p-1} \to X$. It is a p-1-simplex. Think of it as a "face" of T.
Now we can define the boundary homomorphisms.

<u>Def</u> For a sing. p-simplex T define

$$\partial T = \sum_{i=0}^{p} (-1)^i \, T \circ l(e_0, \ldots \hat{e}_i, \ldots e_p) \in S_{p-1}(X).$$

For a sing p-chain $c = \sum n_i T_i$ define

$$\partial c = \sum n_i \partial T_i \in S_{p-1}(X).$$

Then $\partial : S_p(X) \to S_{p-1}(X)$ is a homomorphism.

Def If $f: X \to Y$ is cont. define the induced homomorphism

$$f_\# : S_p(X) \to S_p(Y)$$

by defining it first on sing. p-simplices by

$$f_\#(T) = (f \circ T): \Delta_p \to Y \in S_p(Y),$$

then ~~ex~~ extending to a group homomorphism.

Thm $f_\#$ commutes with ∂.

Pf $\partial f_\#(T) = \partial(f \circ T) = \sum_{p=0}^{p} (-1)^i (f \circ T) \circ l[e_0, \dots, \hat{e}_i, \dots, e_p]$

$= \sum (-1)^i f \circ (T \circ l[e_0, \dots \hat{e}_i \dots e_p])$

$= f_\#\left(\sum (-1)^i T \circ l[e_0, \dots, \hat{e}_i \dots e_p] \right)$

$= f_\# \partial(T),$

for any p-simp. T. This extends to any sing p-chain.

$\square$

Next we show $\partial \partial = 0$.

By definition

$$\partial \ell[a_0, \dots a_p] = \sum_{i=0}^{p} (-1)^i \ell[a_0 \dots a_p] \circ \ell[e_0, \dots, \hat{e}_i, \dots e_p].$$

Claim this $= \sum_{i=0}^{p} (-1)^i \ell[a_0 \dots \hat{a}_i \dots a_p]$.

This is because $\ell[a_1 \dots a_p] \circ \ell[e_0 \dots \hat{e}_i \dots e_p] = \ell[a_0 \dots \hat{a}_i \dots a_p]$.

I'll check this for one case and let you generalize: $p=3$. $i=2$.

$$\ell[a_0 a_1 a_2 a_3] \circ \ell[e_0 e_1 e_3] \stackrel{?}{=} \ell[a_0 a_1 a_3].$$

We just evalute both sides at $(x_1, x_2, 0, 0, \dots)$

$$\ell[e_0 e_1 e_3](x_1, x_2, 0, \dots) = e_0 + x_1(e_1 - e_0) + x_2(e_3 - e_0)$$
$$= (x_1, 0, x_2, 0, 0, \dots)$$
$$\ell[a_0 a_1 a_2 a_3](x_1, 0, x_2, 0, 0, \dots) = a_0 + x_1(a_1 - a_0) + 0(a_2 - a_0) + x_2(a_3 - a_0)$$
$$= [1 - x_1 - x_2]a_0 + x_1 a_1 + x_2 a_3.$$

$$\ell[a_0 a_1 a_3](x_1, x_2, 0 \dots) = a_0 + x_1(a_1 - a_0) + x_2(a_3 - a_0)$$
$$= \text{the same.}$$

One thing to watch out for; the sum in the summation symbol is the group operation, but the sums in the computing above are vector space sums in $\mathbb{R}^\infty$. Do not confound them.

Now $\partial\partial l(a_0 \cdots a_p) = \sum_{i=0}^{p} (-1)^i \partial l(a_0 \cdots \hat{a}_i \cdots a_p)$

$$= \sum_{i=0}^{p} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j l(a_0 \cdots \hat{a}_j \cdots \hat{a}_i \cdots a_p) + \sum_{j=i+1}^{p} (-1)^{j-1} l(a_0 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_p) \right)$$

$= 0$ since $l(a_0 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_p)$ = itself and appears twice with opposite signs.

(See Lemma 5.3, pg 30).

For a general sing. p-simp. $T : \Delta_p \to X$

$$\partial\partial(T) = \partial \sum_{i=0}^{p} (-1)^i T \circ l(\varepsilon_0 \cdots \hat{\varepsilon}_i \cdots \varepsilon_p)$$

$$= \sum (-1)^i T \circ \partial l(\varepsilon_0 \cdots \hat{\varepsilon}_i \cdots \varepsilon_p)$$ [as in the Thm that $f_\# \partial = \partial f_\#$]

$$= \sum (-1)^i T \circ 0 = 0.$$

<u>Def</u> $\mathcal{S}(X) = \{S_p(X), \partial_p\}$. ~~The homology groups~~

It is the <u>singular homology chain complex</u> of X. The homology groups of $\mathcal{S}(X)$ are the <u>singular homology groups</u> of X

$$H_p(X) = \frac{Z_p(X)}{B_p(X)} = \frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}}.$$

Reduced Homology: Let $\varepsilon : S_0(X) \to \mathbb{Z}$ be determined by setting $\varepsilon(T) = 1$ for each sing. 0-simp. and extending. Then replace $\partial : S_0(X) \to 0$ with $\varepsilon : S_0(X) \to \mathbb{Z}$. This makes a new chain complex $\tilde{S}(X)$. This determines the reduced singular homology groups as usual.

Def Let X be a top. sp. If all the reduced sing. hom. gps are trivial, we say X is acyclic.

Facts If $f : X \to Y$ is cont. there are induced hom.'s on the homology gps: $f_* : H_p(X) \to H_p(Y)$ just like before.

1. If $id : X \to X$ is the identity, then $id_* : H_p(X) \to H_p(X)$ is the identity isomorphism.

2. If $h : X \to Y$ is a homeo., then $h_* : H_p(X) \to H_p(Y)$ is an isomorphism.

3. $(g \circ f)_* = g_* \circ f_*$.

4. The relation between sing. homology and reduced ~~simp~~ sing hom. is the same as with the simplicial hom. theory. See Thm 29.4.