Cones in Singular Homology

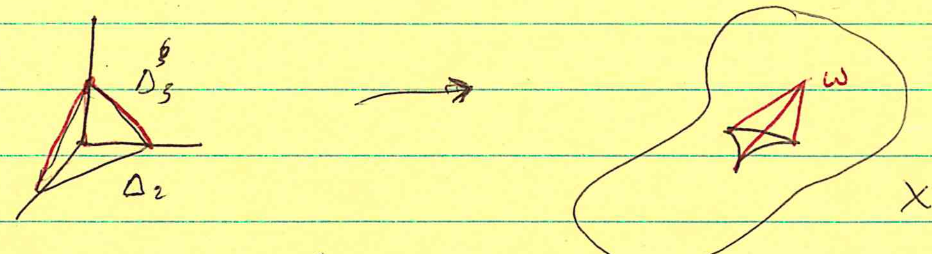
We cannot talk about forming a cone over a general top. sp. but we can do the following.

Def Let $X \subset \mathbb{R}^\infty$. Then X is <u>star convex</u> relative to $w \in X$ if for each $x \in X - \{w\}$ the line seg from x to w lies within X.

Def Let X be star convex relative to w. ($w \in X \subset \mathbb{R}^\infty$.) Let $\sigma : \Delta_p \to X$ be a sing. p-simp. Define

$$[\sigma, w] : \Delta_{p+1} \to X$$

to be the sing $p+1$-simp. given by letting it take e_{p+1} to w and the line segments in Δ_{p+1} from points $x \in \Delta_p$ to e_{p+1}, linearly to the line segment from $\sigma(x)$ to w.



For a p-chain $c = \sum n_i \sigma_i$ in $S_p(X)$ define

$$[c, w] = [\sum n_i \sigma_i, w] = \sum n_i [\sigma_i, w] \in S_{p+1}(X).$$

Notice that $[\ell[a_0 \cdots a_p], w] = \ell[a_0 \cdots a_p w]$. (You check this.)

The textbook shows $[\sigma, w]$ is in fact continuous.

Next we compute a boundary formula.

Lemma 29.5 Let X be star convex wrt w. Let $c \in S_p(X)$. Then

$$\partial [c, w] = \begin{cases} [\partial c, w] + (-1)^{p+1} c & p > 0 \\ \varepsilon(c) T_w - c & p = 0 \end{cases}$$

where $T_w : \Delta_0 \to \{w\}$ and $\varepsilon(c) = \varepsilon(\sum n_i \sigma_i) = \sum n_i$.

Pf The proof is a calcation. See textbook. Compare with formulas $(*)$ on pg 45. The proofs for these are in the class notes.

Thm 29.6 Star convex spaces are acyclic.

Pf See textbook. It is short and easy.

Notes: Many types of spaces can be embedded into $\mathbb{R}^J$. See Munkres' "Topology" §34 and §36.

Fact Sing. simplicies are acyclic.