§30 Axioms for Singular Homology Theory.

Once we define homology of pairs, most of the axioms are easy the verify. However, the homotopy and excision axioms are harder.

Def Let X be a top. sp. and let $A \subset X$ be a subspace. We can regard $S_p(A)$ as a subgroup of $S_p(X)$. Then we define

$$S_p(X,A) = S_p(X) / S_p(A).$$

It is clear that $S_p(X,A)$ is a free group we can take the standard generators of $S_p(A)$ to be generators of $S_p(X)$.

The restriction of $\partial : S_p(X) \to S_{p-1}(X)$ to $S_p(A)$ has its image in $S_{p-1}(A)$. Hence it induces a ~~map~~ homomorphism

$$\partial : S_p(X,A) \to S_{p-1}(X,A).$$

It is easy to check $\partial\partial = 0$. Let $\mathcal{S}(X,A) = \{S_p(X,A), \partial_p\}_{\forall p}$. Its homology groups

$$H_p(X,A)$$

are the relative sing. homology groups of the pair (X,A).

If $f:(X,A)\to(Y,B)$ is a cont. map then

$$f_\#: S_p(X) \to S_p(Y)$$

when restricted to $S_p(A)$ has its image in $S_p(B)$, and thus induces a hom.

$$f_\#: S_p(X,A) \to S_p(Y,B).$$

You can check $f_\#$ still commutes with ∂, so there is an induced hom. on the homology gps:

$$f_*: H_p(X,A) \to H_p(Y,B).$$

The Axioms

(You may want to have them on ~~hard~~ hand.)

Axioms 1 and 2 are covered by Thm 30.1 whose proof you should do.

Axiom 4 is Thm 30.2, whose proof you can read. Axiom 3 is mentioned in the proof of Thm 30.2, but not by name ("The naturality of ∂_* ...").

Axiom 7 is Thm 30.3. The proof is not hard.

Axiom 8 is Thm 30.4. The proof is not hard.

This leaves Axioms 5 and 6.