Axiom 5 (Homotopy Axiom) for Sing. Hom.

(a) Let $f, g : X \to Y$ be cont. If $f \simeq g$ (f is homotopic to g), then $f_* = g_* : H_p(X) \to H_p(Y)$.

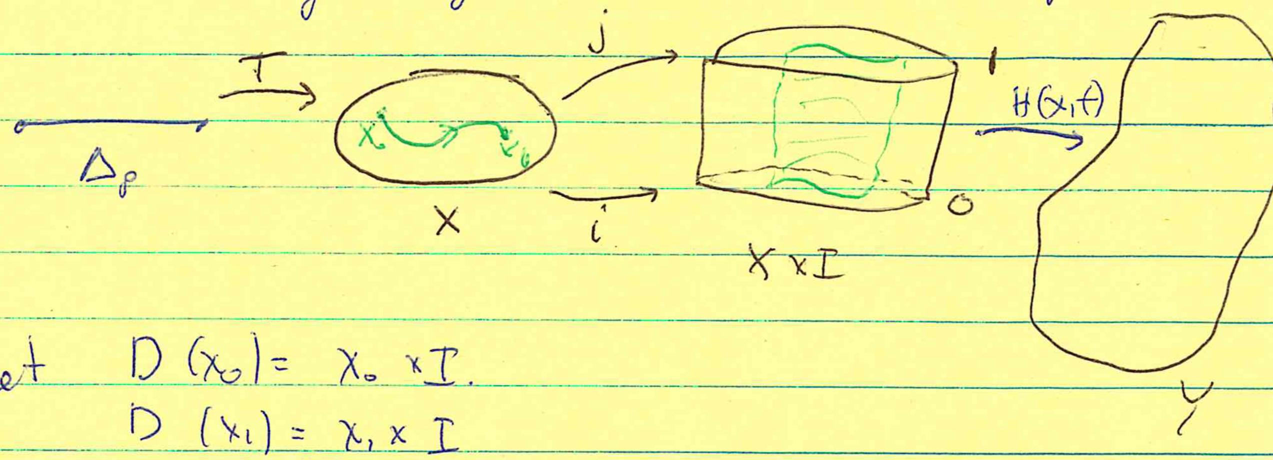
(b) Let $f, g\ (X, A) \to (Y, B)$ be cont. ($f(A) \subset B, g(A) \subset B$) They are homotopy as maps on pairs, ~~or homotopic relative to B,~~ if $\exists\ H : X \times I \to Y$ that is cont., s.t.

$$H(x,0) = f(x) \qquad \forall\ x \in X$$
$$H(x,1) = g(x) \qquad \forall\ x \in \cancel{Y}$$
$$H(a,t) \in B \qquad \forall\ a \in A,\ t \in [0,1] = I.$$

If this holds Axiom 5 requires

$$f_* = g_* : H_p(X, A) \to H_p(\cancel{X}, B).$$

Idea To prove Axiom 5 for sing hom we use the following (rough) idea. Pictures are for $p=1$.



Let $D(x_0) = x_0 \times I$.

$D(x_1) = x_1 \times I$

$D(T(\Delta_p)) = T(\Delta_p) \times I$. $\quad \partial T = x_1 - x_0$

$$\partial D(T) = x_0 \times I + j(T) - x_1 \times I - i(T)$$

$$\partial D(T) = j(T) - i(T) - D(\partial T)$$

$$\partial D(T) + D(\partial T) = j(T) - i(T)$$

(Compare with definition on page 66.)

Lemma 30.6 shows such a $D: S_p(X) \to S_{p+1}(X \times I)$ exists in general. Then Thm 30.7 shows that if $f \simeq g$, then $f_\#$ is chain homotopic to $g_\#$. Finally, Thm 12.4 $\Rightarrow f_* = g_*$.

Lemma 30.6 $\forall$ top. sp. X and $p \geq 0$ $\exists$ a homomorphism

$$D_X : S_p(X) \longrightarrow S_{p+1}(X \times I) \quad \text{s.t.}$$

ⓐ If $\sigma : \Delta_p \to X$ is a sing. simplex, then

$$\partial D_X \sigma + D_X \partial \sigma = j_\#(\sigma) - i_\#(\sigma),$$

where $i : X \to X \times \{0\} \subset X \times I$, $j : X \to X \times \{1\} \subset X \times I$ are inclusion maps.

ⓑ If $f : X \to Y$ is continuous, then

$$\begin{tikzcd} S_p(X) \arrow[r, "D_X"] \arrow[d, "f_\#"] & S_{p+1}(X \times I) \arrow[d, "(f \times id)_\#"] \\ S_p(Y) \arrow[r, "D_Y"] & S_{p+1}(Y \times I) \end{tikzcd}$$

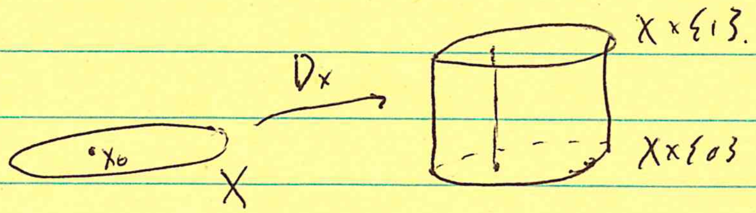
commutes (i.e. D is natural).

Pf: Use induction on p. Let $p = 0$. Let $x_0 = \sigma(\Delta_0)$.
Define $(D_X \sigma) : \Delta_1 \to X \times I$ by

$$D_X \sigma(t, 0, 0, \ldots) = (x_0, t) \in X \times I.$$

Δ_1 $0 \leq t \leq 1$.

We can extend D_X to 0-chains, $S_0(X)$.



(a) $\partial D_X \sigma(\Delta_0) = [x_0 \times \{1\}] - [x \times \{0\}] = j_\#(\sigma) - i_\#(\sigma)$

and $D_X \partial(\Delta_0) = D_X 0 = 0$. So, (a) holds.

(b) This is clear. Suppose $f: X \to Y$ take x_0 to y_0.

$(f \times id)_\# \circ D_X(x_0) = f(x_0) \times I = y_0 \times I = D_Y(f_\#(x))$.

Now suppose D_X is defined for dimensions $< p$ and satisfies (a) and (b), for all X.

Step 1 Suppose $X = \Delta_p$. Let $\sigma = id: \Delta_p \to \Delta_p$.
We shall define $D_X(\sigma) \in S_{p+1}(\Delta_p \times I)$.

Let $i, j: \Delta_p \to \Delta_p \times I$ be $i(x) = (x, 0)$ and $j(x) = (x, 1)$.

Let $c = j_\#(\sigma) - i_\#(\sigma) - \underbrace{D(\partial\sigma)}_{\text{defined since dim } p-1} \in S_p(\Delta_p \times I)$

Claim c is a cycle:

$\partial c = \partial j(\sigma) - \partial i(\sigma) - \partial D(\partial\sigma) =$
$\partial j(\sigma) - \partial i(\sigma) - j(\partial\sigma) - i(\partial\sigma) + D\partial\partial\sigma = 0$
since ∂ commutes with i and j.

Since $\Delta_p \times I$ is convex it is acyclic (by Thm 29.6).
Thus, $\exists\ d \in S_{p+1}(\Delta_p \times I)$ s.t. $\partial d = c$.
Let $D_{\Delta_p}(\sigma) = d$.

(a) is an immediate calculation.

[(b) is not relevant (yet) since it involves two spaces and a cont func; $f: X \to Y$, and we are only looking at Δ_p]

Step 2 Let X be a top. sp. Let $\tau : \Delta_p \to X$ be a sing p-smp. Define

$$D_X \tau = (\tau \times id_I)_\# (D_{\Delta_p} \sigma).$$

Extend D_X to $S_p(X)$.

$$\partial D_X \tau = (\tau \times id_I)_\# \partial D_{\Delta_p}(\sigma)$$

$$= (\tau \times id_I)_\# (j_\# \sigma - i_\# \sigma - D_{\Delta_p}(\partial \sigma))$$

$$= j_\#(\tau) - i_\#(\tau) - (\tau \times id_I) D_{\Delta_p}(\partial \sigma)$$

since: $(\tau \times id_I)(j_\# \sigma) = \tau \times id_I(\Delta_p \times \{1\}) = \tau(\Delta_p) \times \{1\}$
$= j_\#(\tau(\Delta_p))$.

likewise for $(\tau \times id_I) i_\#(\sigma)$.

We need to show that $(\tau \times id_I)_\# D_{\Delta_p}(\partial \sigma) = D_X(\partial \tau)$ to show that (a) holds. Here we will use that (b) holds in dim $p-1$.

~~Thus we ~~can~~ have defined D_X on sing. p-simplices $\forall p \geq 0$. Extend homomorphically to $S_p(X)$. $\square$~~

$$\begin{array}{ccc} \partial\sigma \in S_{p-1}(\Delta_p) & \xrightarrow{D_{\Delta_p}} & S_p(\Delta_p \times I) \\ \tau_\# \downarrow & & \downarrow (\tau \times id_I)_\# \\ S_{p-1}(X) & \xrightarrow{D_X} & S_p(X \times I) \end{array}$$

commutes. Thus,

$$(\tau \times id_I)_\# D_{\Delta_p}(\partial\sigma) = D_X \tau_\#(\partial\sigma) = D_X \partial \tau_\#(\sigma) =$$

$$D_X \partial(\tau \circ \sigma) = D_X \partial \tau, \text{ since } \sigma = id \text{ on } \Delta_p.$$

Step 3 Now we check (b). Let $\tau \in S_p(X)$ be a generator $\tau : \Delta_p \to X$. Let $f: X \to Y$.

Then $(f \circ \tau) \times id_I = (f \times id_I) \circ (\tau \times id_I)$.

$$D_Y(f_\# \tau) = D_Y(f \circ \tau) = \left((f \circ \tau) \times id_I\right)_\# D_{\Delta_p}(\sigma)$$

$$= [f \times id_I \circ \tau \times id_I]_\# D_{\Delta_p}(\sigma) =$$

$$(f \times id_I)_\# (\tau \times id_I)_\# D_{\Delta_p}(\sigma) = (f \times id_I)_\# D_X \tau.$$

Thus (b) holds.

Thus we have defined D_X on sing. p-simplicies $\forall p \geq 0$. Extend homomorphically to $S_p(X)$. $\square$

Thm 30.7 If $f, g : (X, A) \to (Y, B)$ are homotopic, then $f_* = g_*$. The same holds in reduced sing homology if $A = B = \phi$.

Pf (non reduced case only). Let $F : (X \times I, A \times I) \to (Y, B)$ be the homotopy:

$$F(x, 0) = f(x)$$
$$F(x, 1) = g(x)$$
$$F(a, t) \in B \quad \forall a \in A, t \in I.$$

It induces a hom. $F_\# : S_{p+1}(X \times I, A \times I) \to S_{p+1}(Y, B)$.

Let $i : (X, A) \to (X \times I, A \times I)$ be $i(x) = x \times \{0\} = (x, 0)$.

Let $j : (X, A) \to (X \times I, A \times I)$ be $j(x) = x \times \{1\} = (x, 1)$.

Let $D_X : S_p(X) \to S_{p+1}(X \times I)$ be as in the lemma.

Let $D_A : S_p(A) \to S_{p+1}(A \times I)$ be as in the lemma.

Let $k : A \to X$ be inclusion.

Then the following diagram commutes.

$$\begin{array}{ccc} S_p(A) & \xrightarrow{D_A} & S_{p+1}(A \times I) \\ {\scriptstyle k_\#}\downarrow & & \downarrow{\scriptstyle (k \times Id_I)_\#} \\ S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \end{array}$$

Thus $D_X|_{S_p(A)}(S_p(A)) \subset S_{p+1}(A \times I)$.

Therefore D_X induces a homomorphism

$$D_{X,A} : \frac{S_p(X)}{S_p(A)} \longrightarrow \frac{S_{p+1}(X \times I)}{S_{p+1}(A \times I)} \quad \text{or}$$

$$D_{X,A} : S_p(X,A) \to S_{p+1}(X \times I, A \times I).$$

Formula ⓐ $\partial D + D\partial = j_\# - i_\#$ holds for $D_{X,A}$ because it holds for D_X : if $\gamma(\Delta_p) \subset A$ ⓐ becomes $0=0$; if $\gamma(\Delta_p)$ is not if A ⓐ holds b/s $D_{X,A} = D_X$.

Define $D = F_\# \circ D_{X,A} : S_p(X,A) \to S_{p+1}(Y,B)$

$\uparrow$ F was the homotopy.

Then

$$\partial D = \partial F_\# \circ D_{X,A} = F_\# \partial D_{X,A} = F_\#(j_\# - i_\# - D_{X,A}\partial)$$

$$= (F \circ j)_\# - (F \circ i)_\# - F_\#(D_{X,A}\partial) = f_\# - g_\# - D\partial.$$

The result follows from Thm 12.4. $\blacksquare$