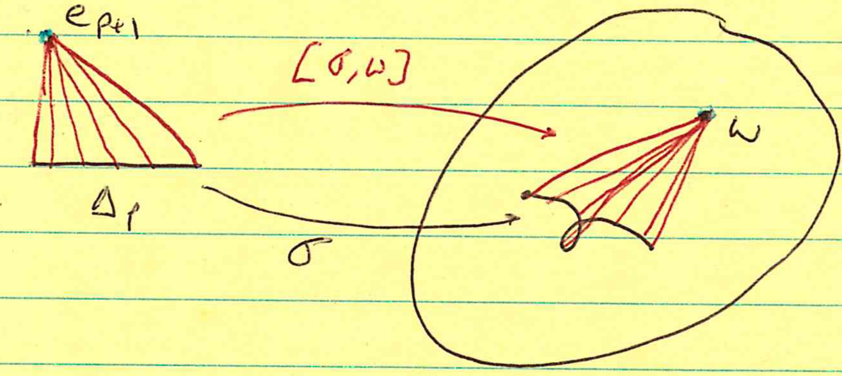
§31 Excision Axiom in Sing. Homology

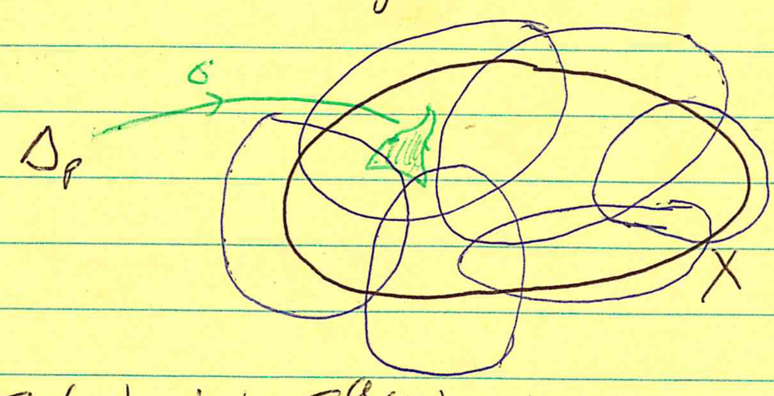
Recall Let $X \subset \mathbb{R}^\infty$ be a star convex top sp. wrt $w \in X$. Let $\sigma : \Delta_p \to X - \{w\}$. Then $[\sigma, w] : \Delta_{p+1} \to X$ defined by

$$[\sigma, w](x) = \begin{cases} \sigma(x) & , x \in \Delta_p \\ w & x = e_{p+1} \\ \text{linear extension} & \text{otherwise.} \end{cases}$$

(See Definition 2 pg 165).



Def Let X be a top. sp. and let $\mathcal{U}$ be a collection of subsets of X whose interiors cover X. A singular simplex of X, $\sigma : \Delta_p \to X$, is **$\mathcal{U}$-small** if its image lies inside an element of $\mathcal{U}$.



Given $S_p(X)$ let $S_p^{\mathcal{U}}(X)$ be the subgp generated by $\mathcal{U}$-small singular p-simplices.

Def We defined a <u>subdivision operator</u> similar to the one defined in §16. Let X be a top. sp. We will define a homomorphism $sd_X : S_p(X) \to S_p(X)$, $\forall p$, as follows.

<u>$p=0$</u> For $p=0$ let $sd_X : S_0(X) \to S_0(X)$ be the identity.

<u>$p=1$</u> We define $sd_X : S_1(X) \to S_1(X)$ in two steps. First, we use $X = \Delta_1$.

(i) Let $id : \Delta_1 \to \Delta_1$ be the identity.
Let $\hat{\Delta}_1 = (\frac{1}{2}, 0, 0, \dots)$, the midpt of Δ_1.

Let $sd_{\Delta_1}(id) = -[sd_{\Delta_0}(\partial\, id), \hat{\Delta}_1]$.

We unpack this: $\partial(id) = id \circ l(e_1) - id \circ l(e_0)$.

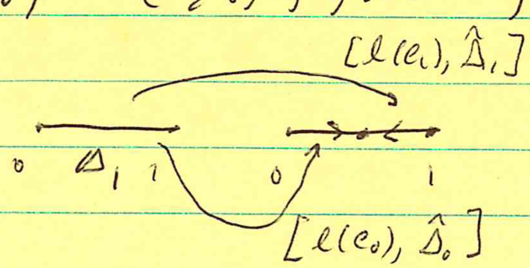
$[l(e_1), \hat{\Delta}_1]$ is a map $\Delta_1 \to \mathbb{R}^\infty$ given by

$e_1 + t(\hat{\Delta}_1 - e_1) = (1 - \frac{1}{2}t, 0, 0 \dots)$ (see pg 162)

$[l(e_0), \hat{\Delta}_1] : \Delta_1 \to \mathbb{R}^\infty$ is given by

$e_0 + t(\hat{\Delta}_1 - e_0) = (\frac{1}{2}t, 0\ 0 \dots)$

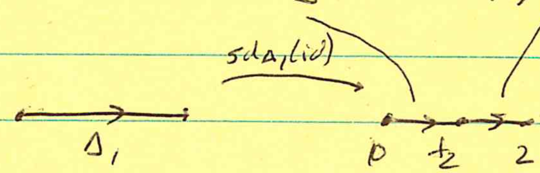
In pictures



Now,

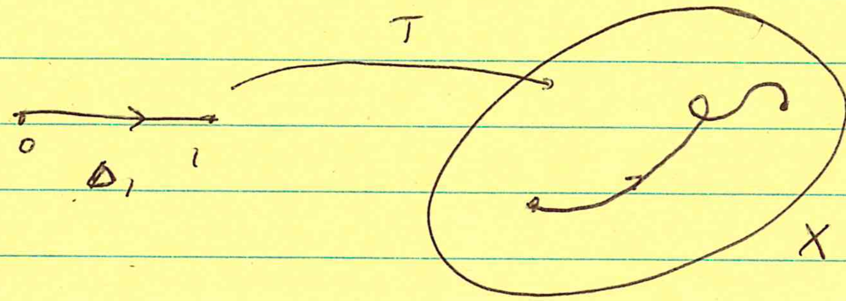
$$-[\, sd_{\Delta_0}(\partial\, id), \hat{\Delta}_1 \,] = -\left([\, l(e_1), \hat{\Delta}_1 \,] - [\, l(e_0), \hat{\Delta}_1 \,] \right)$$

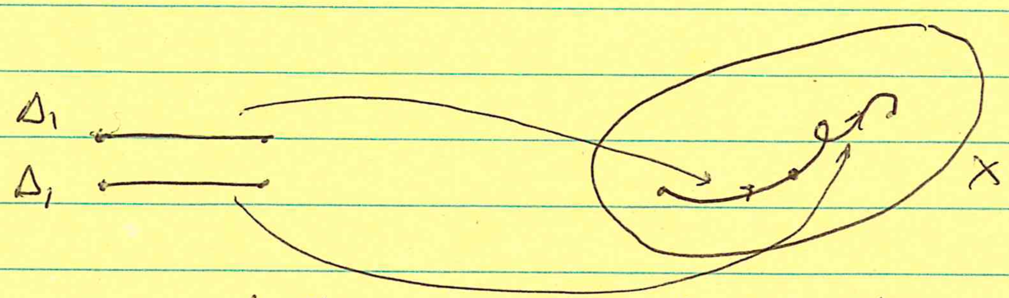
$$= [\, l(e_0), \hat{\Delta}_1 \,] + [\, \hat{\Delta}_1 , l(e_1) \,]$$



(ii) Let $T : \Delta_1 \to X$ be any sing. 1-simplex.

Let $sd_X T = T_{\#}(sd_{\Delta_1}(id))$

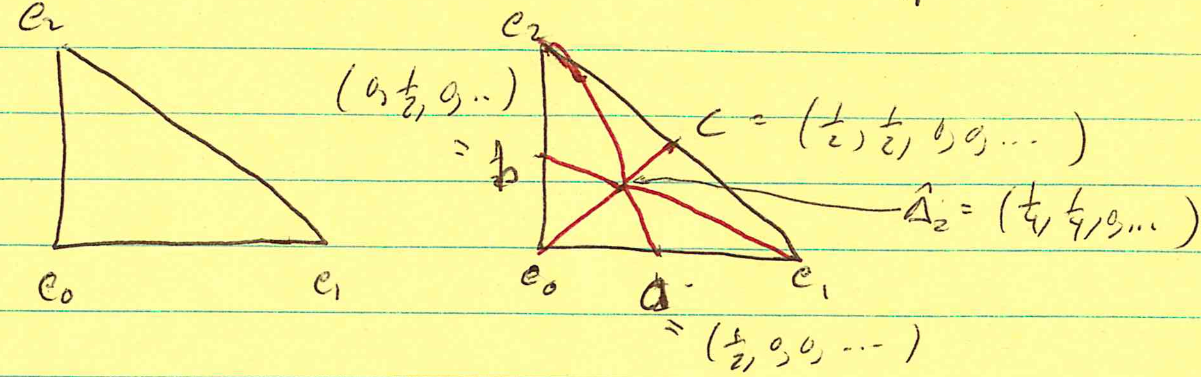




$sd_X T$ is the sum of these two 1-simplices.

P-2 (i) Let $id : \Delta_2 \to \Delta_2$ be the identity.

$sd_{\Delta_2}(id)$ will be the sum of six maps.



They are $\ell(e_0, a, \hat{\Delta}_2)$, $\ell(a, e_1, \hat{\Delta}_2)$, $\ell(e_1, c, \hat{\Delta}_2)$
$\ell(c, e_2, \hat{\Delta}_2)$, $\ell(e_2, b, \hat{\Delta}_2)$, $\ell(b, e_0, \hat{\Delta}_2)$.

We define
$$sd_{\Delta_2}(id) = [\, sd_{\Delta_1}(\partial\, id), \hat{\Delta}_2 \,].$$

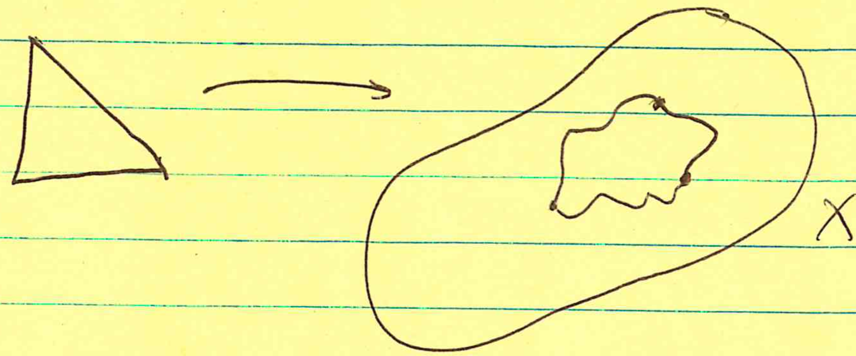
Now $id = \ell(e_0, e_1, e_2)$. $\partial(id) = \ell(e_1 e_2) - \ell(e_0, e_2) + \ell(e_0, e_1)$.
Then $sd\Delta_1$ splits each of these in two:

$$sd_{\Delta_1}(\ell(e_1 e_2)) = \ell(e_1, c) + \ell(c, e_2)$$
$$sd_{\Delta_1}(\ell(e_0, e_2)) = \ell(e_0, b) + \ell(b, e_2)$$
$$sd_{\Delta_1}(\ell(e_0, e_1)) = \ell(e_0, a) + \ell(a, e_1)$$

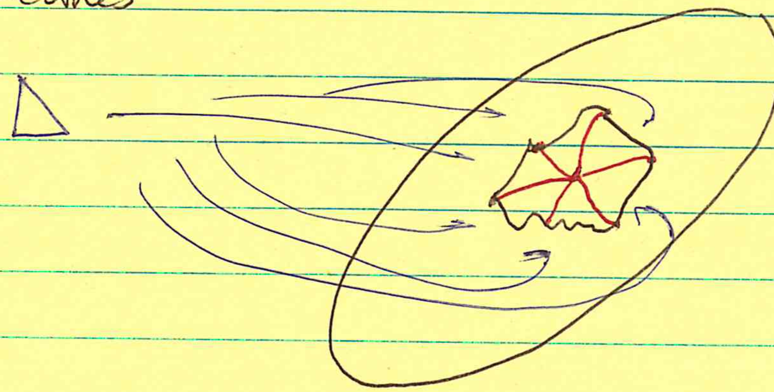
Then $[\, sd_{\Delta_1}(\partial\, id), \hat{\Delta}_2 \,] =$
$\ell(e_1\, c\, \hat{\Delta}_2) + \ell(c, e_2, \hat{\Delta}_2) - \ell(e_0, b, \hat{\Delta}_2) + \ell(b, e_2, \hat{\Delta}_2)$
$+ \ell(e_0, a, \hat{\Delta}_2) + \ell(a, e_1, \hat{\Delta}_2) =$

$\ell(e_0, a, \hat{\Delta}_2) + \ell(a, e_1, \hat{\Delta}_2) + \ell(e_1, c, \hat{\Delta}_2) + \ell(c, e_2, \hat{\Delta}_2) +$
$\ell(e_2, b, \hat{\Delta}_2) + \ell(b, e_0, \hat{\Delta}_2)$, as desired.

(ii) Let $T: \Delta_2 \to X$. Define $sd_X(T) = T_\#(sd_{\Delta_2}(id))$



becomes



the sum of these six maps.

$\underline{p>2}$ Continue inductively.

$$sd_{\Delta_p}(id) = (-1)^p \left[sd_{\Delta_{p-1}}(\partial \underset{id:\Delta_p \to \Delta_p}{id}), \underbrace{\hat{\Delta}_p}_{\text{barycenter of } \Delta_p} \right]$$

$$sd_X(T) = T_\#(sd_{\Delta_p}(id)).$$

(See Lemma 31.2)

Lem 31.1 The homomorphism $sd_X : S_p(X) \to S_p(X)$ is
① An augmentation-preserving chain map,
② it is natural: if $f : X \to Y$ is cont, then
$$f_\# \circ sd_X = sd_Y \circ f_\# .$$

Partial Proof Recall $\varepsilon : S_0(X) \to \mathbb{Z}$ ~~is~~ given by sending each sing 0-simp. to 1, and extending, is the augmentation map. Aug.-pres. means $\varepsilon \circ sd_X = \varepsilon$. See pg 72. But sd_X is the identity on $S_0(X)$, so this is obvious. Naturality holds in dim zero for the same reason.

For positive p compute: $\overset{f : X \to Y}{T}$ is a gen. of $S_p(X)$

$$f_\#(sd_X T) = f_\# T_\# (sd_{\Delta_p}(id)) = (f \circ T)_\# (sd_{\Delta_p}(id))$$

$$= sd_Y (f \circ T) = sd_Y (f_\# T) \;, \text{ since } f \circ T : \Delta_p \to Y.$$

This extends to all of $S_p(X)$.

To show sd_X is a chain map we need to show $sd_X \circ \partial = \partial \circ sd_X$. See textbook. $\square$

Thm 31.3 Let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Given $\tau : \Delta_p \to X$, $\exists m$ s.t. each term of sd_X^m is $\mathcal{A}$-small.

Pf See textbook. Pf uses Lebesgue number of an open cover of Δ_p (which is compact) given by $\{\tau^{-1}(\underset{\text{interior}}{A}) \mid A \in \mathcal{A}\}$. See Section 27 of Munkres' "Topology" (Math 530).

Lem 31.4 Let m be given. $\forall$ top sp. X, $\exists$ a homomorphism

$$D_X : S_p(X) \to S_{p+1}(X) \text{ s.t. } \forall \tau : \Delta_p \to X$$

$$\partial D_X \tau + D_X \partial \tau = sd^m \tau - \tau \qquad (*)$$

and $\quad f : X \to Y$ cont $\Rightarrow$

$$f_\# \circ D_X = D_Y \circ f_\# \qquad (**) \quad (D_X \text{ is natural})$$

Note: D_X is a chain homotopy (§12) from sd^m to the id.

Pf $(p=0)$. For a sing. 0-simp. $\tau : \Delta_0 \to X$ define $D_X \tau = 0$. $(*)$ and $(**)$ follow.

$(p>0)$ Suppose D_X is defined for $\dim < p$ with $(*)$ and $(**)$ true.

(i) First define $D_X \sigma$ for a special case, $X = \Delta_p$, $\sigma = id : \Delta_p \to \Delta_p$. Consider

$$c_p = sd^m(id) - id - D_{\Delta_p}(\partial\, id) \in S_p(\Delta_p).$$

Claim c_p is a cycle.

$$\partial c_p = \partial\, sd^m id - \partial\, id - \partial\, D_{\Delta_p}(\partial\, id) =$$

$$\partial\, sd^m id - \partial\, id - (sd^m \partial\, id - \partial\, id - D \partial\partial\, id) = 0.$$

Since Δ_p is acyclic, $\exists\ d_{p+1} \in S_{p+1}(\Delta_p)$ with $\partial d_{p+1} = c_p$.

Let $D_{\Delta_p}(id) = d_{p+1}$. You can check ☆ holds.

(ii) Let $\sigma : \Delta_p \to X$ and define

$$D_X \sigma = \sigma_\# (D_{\Delta_p}(id)).$$

Then ☆ holds : easy see textbook.
For ☆☆, let $f : X \to Y$ be cont.

$$\begin{array}{ccc} \sigma \in S_p(X) & \xrightarrow{D_X} & S_{p+1}(X) \\ f_\# \downarrow & & \downarrow f_\# \\ S_p(Y) & \xrightarrow[D_Y]{} & S_{p+1}(Y) \end{array}.$$

$$f_\# D_X(\sigma) = f_\# \sigma_\# (D_{\Delta_p}(id)) = (f \circ \sigma)_\# D_{\Delta_p}(id) = D_Y(f \circ \sigma)$$
$$= D_Y f_\# (\sigma). \quad \square$$

Def Let X be a top. sp. Let $\mathcal{U}$ be a collection of subsets whose interiors cover X. Let $S_p^{\mathcal{U}}(X)$ be the subgp of $S_p(X)$ generated by $\mathcal{U}$-small sing p-simplices.

Let $\mathcal{S}_p(X)$ be the chain complex $\{S_p(X), \partial\}$.

Let $\mathcal{S}_p^{\mathcal{U}}(X)$ be the chain complex $\{S_p^{\mathcal{U}}(X), \partial\}$.

Let $\mathcal{S}^{/\mathcal{U}}(X)$ " " " " $\left\{\frac{S_p(X)}{S_p^{\mathcal{U}}(X)}, \partial\right\}$.

Notes $S_0^{\mathcal{U}}(X) = S_0(X)$ and ε defines an aug-pres. chain map for $\mathcal{S}^{\mathcal{U}}(X)$, so reduced homology gps are defined.

Let $A \subset X$. $sd^m : S_p(A) \to S_p(A)$ and $D_X : S_p(A) \to S_{p+1}(A)$. Thus, they induce a chain map and a chain homotopy on the relative chain complex $\mathcal{S}(X, A)$. D_X is a chain homotopy of sd^m to id on $\mathcal{S}(X, A)$.

Both sd^m and D_X take $\mathcal{S}^{\mathcal{U}}(X)$ into itself.

Thm 31.5 Let X be a top. sp. Let $\mathcal{A}$ be a collection of subsets whose interiors cover X. Then the inclusion ~~homomorphism~~ map $\mathcal{S}^{\mathcal{A}}(X) \to \mathcal{S}(X)$ induces an isomorphism in homology, both ordinary and reduced.

Pf Consider the short exact seq. of chain complexes

$$0 \to S_p^{\mathcal{A}}(X) \xrightarrow{i} S_p(X) \xrightarrow{\pi} S_p^{\not{\mathcal{A}}}(X) \to 0.$$

By the zig-zag lemma we have the long exact seq

$$\cdots \to H_{p+1}^{\not{\mathcal{A}}}(X) \to H_p^{\mathcal{A}}(X) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p^{\not{\mathcal{A}}}(X) \to H_{p-1}^{\mathcal{A}}(X) \to \cdots$$

Claim $H_*^{\not{\mathcal{A}}}(X) = 0$. This $\Rightarrow i_*$ is a ISO.

Let $\alpha \in H_p^{\not{\mathcal{A}}}(X)$. Let $\alpha = [c_p]$. So, $c_p \in S_p(X)$ with $\partial c_p \in S_{p-1}^{\mathcal{A}}(X)$. We will show that c_p bounds, i.e. $\exists\, d_{p+1} \in S_{p+1}(X)$ s.t. $c_p - \partial d_{p+1} \in S_p^{\mathcal{A}}(X)$.

Choose m s.t. each simplex of $sd^m c_p$ is $\mathcal{A}$-small.

Let D_X be the chain homotopy of Lem 31.4.
Let $d_{p+1} = -D_X c_p$.

$$c_p - \partial d_{p+1} = c_p + \partial D_X c_p = c_p - D_X \partial c_p + sd^m c_p - c_p$$

$$= -D_X \partial c_p + sd^m c_p \in S_p^{\mathcal{A}}(X). \quad \blacksquare$$

Cor 31.6 Let $B \subset X$. Let $S_p^{a}(X,B) = \dfrac{S_p^{a}(X)}{S_p(B)}$.

Then the inclusion map $S_p^{a}(X,B) \to S_p(X,B)$ induces an iso. of homology groups.

Pf Consider

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}(B) \arrow[r, "i"] & \mathcal{S}(X) \arrow[r, "\pi"] & \mathcal{S}(X,B) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{S}^{a}(B) \arrow[r, "i"] \arrow[u] & \mathcal{S}^{a}(X) \arrow[r, "\pi"] \arrow[u] & \mathcal{S}^{a}(X,B) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$\leftarrow$ inclusion maps.

Then we have the long exact seq's,

$$\begin{tikzcd}
\arrow[r] & H_p(B) \arrow[r] & H_p(X) \arrow[r] & H_p(X,B) \arrow[r] & H_{p-1}(B) \arrow[r] & H_{p-1}(X) \arrow[r] & \cdots \\
\arrow[r] & H_p^{a}(B) \arrow[r] \arrow[u, equal] & H_p^{a}(X) \arrow[r] \arrow[u, equal] & H_p^{a}(X,B) \arrow[r] \arrow[u, "\alpha"] & H_{p-1}^{a}(B) \arrow[r] \arrow[u, equal] & H_{p-1}^{a}(X) \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd}$$

Use the five lemma (pg 140) to get that α must be an iso.

$\blacksquare$

Thm 31.7 (Excision, finally!) Let $A \subset X$. If $\mathcal{U}$ is a subset of X s.t. $\bar{\mathcal{U}} \subset \operatorname{int} A$, then

$$H_p(X-\mathcal{U}, A-\mathcal{U}) \cong H_p(X, A).$$

pf Let $j: (X-\mathcal{U}, A-\mathcal{U}) \to (X, A)$ be inclusion.

Let $\mathcal{A} = \{X-\mathcal{U}, A\}$. The interiors of $\mathcal{A}$ cover X.

Consider the homomorphisms induced by inclusion

$$\frac{S_p(X-\mathcal{U})}{S_p(A-\mathcal{U})} \longrightarrow \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)} \longrightarrow \frac{S_p(X)}{S_p(A)}$$

The first is well defined b/c of the choice of $\mathcal{A}$. The second induces a homology iso. by Cor 31.6. We will show the first is an iso. already so it induces a homology iso also. This will give the final result.

Let $\phi: S_p(X-\mathcal{U}) \to S_p^{\mathcal{A}}(X)/S_p^{\mathcal{A}}(A)$ be induced by the inclusion $X-\mathcal{U} \to X$.

<u>Claim:</u> ϕ is onto. Pf: Let $c_p \in S_p^{\mathcal{A}}(X)$. Let $c_p = a_p + b_p$ where a_p is carried by A and $b_p = c_p - a_p$. Then $b_p \in S_p(X-\mathcal{U})$ and $\phi(b_p) = b_p + S_p^{\mathcal{A}}(A)$ which contains c_p.

Claim $\ker \phi = S_p(\cancel{A} \mathcal{U})$. Pf: $\ker\phi = \mathcal{U}$-small p-chains carried by A $= S_p(X-\mathcal{U}) \cap S_p^{\mathcal{U}}(A)$.

But $S_p^{\mathcal{U}}(A) = S_p(A)$ and

$$S_p(X-\mathcal{U}) \cap S_p(A) = S_p((X-\mathcal{U})\cap A) = S_p(A-\mathcal{U}). \quad \square$$