§34 <u>The isomorphism between simplicial and singular homology</u>

<u>Goal</u> Let K be a simplicial complex. Let $X = |K|$. Then

$$\underset{\uparrow \text{simp.}}{H_p(K)} \cong \underset{\uparrow \text{sing}}{H_p(X)}$$

The isomorphism commutes with homomorphisms induced by simplicial maps. This means if $f: K \to L$ is a simplicial map and $g: |K| \to |L|$ is the continuous map it determines (see Lemma 2.7 on pg 12) then

$$\begin{array}{ccc} H_p(K) & \cong & H_p(|K|) \\ f_* \downarrow & & \downarrow g_* \\ H_p(L) & \cong & H_p(|L|) \end{array}$$ commutes.

The iso. also commutes with ∂_*, the connecting homomorphism in the long exact seq of a pair. Everything goes through for reduced and relative homology.

The tool we will use is the ordered hom. thy of §13. There we showed simp. and ordered hom. thy's of simplicial complexes where isomorphic and the iso. commuted with ~~hom~~ induced hom's. You should review that.

Recall An ordered p-simplex is a $p+1$ tuple $(v_0, \dots v_p)$ of verticies of K (not neccessarily distinct) that span a simplex of K. $C'_p(K)$ is generated by these.

$$\partial'(v_0, \dots v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

$$\varepsilon'(v) = 1$$

Extend $\partial' : C'_p(K) \to C'_{p-1}(K)$ and $\varepsilon' : C'_0(K) \to \mathbb{Z}$.

Def Define $\theta : C'_p(K) \to S_p(X)$ by extending

$$\theta((v_0, \dots v_p)) = l(v_0, \dots, v_p).$$

Note θ commutes with ∂', ∂ and ε', ε:

$$\begin{tikzcd} C'_p(K) \arrow[r, "\partial'"] \arrow[d, "\theta"'] & C'_{p-1}(K) \arrow[d, "\theta"'] \\ S_p(X) \arrow[r, "\partial"] & S_{p-1}(X) \end{tikzcd} \qquad \begin{tikzcd} C'_0(K) \arrow[r, "\varepsilon'"] \arrow[d, "\theta"'] & \mathbb{Z} \arrow[d, "id"] \\ S_0(X) \arrow[r, "\varepsilon"] & \mathbb{Z} \end{tikzcd}$$

commute. Thus, θ is an augmentation pres. chain map.

Since θ also commutes with inclusion maps, θ induces a chain map on relative chains

$$\theta : C'_p(K, K_0) \to S_p(X, X_0)$$

where K_0 is a subcomplex of K, $|K_0| = X_0 \subset X$.

Now θ induces hom's on the homology gps, reduced and relative. We will show these are isomorphisms. They are natural. The iso. pf is two Lemmas.

Lemma 34.1 Let $\psi: \mathcal{C} \to \mathcal{C}'$ be a chain map of augmented chain complexes. Then ψ_* is an iso. in reduced hom. iff it is an iso. in ordinary homology.

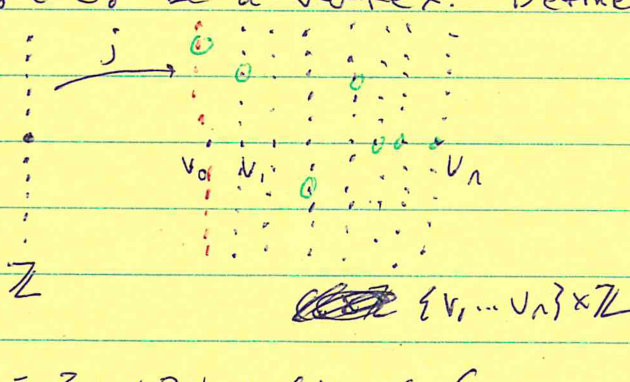
Note: Review Thms 7.2 and Exercise 7.1.

Pf: Consider

$$\begin{array}{ccccccccc} 0 & \to & \ker \varepsilon & \to & C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} & \to & 0 \\ & & \downarrow \psi & & \downarrow \psi & & \downarrow id & & \\ 0 & \to & \ker \varepsilon' & \to & C_0' & \xrightarrow[\varepsilon']{} & \mathbb{Z} & \to & 0 \end{array}$$

The top and bottom sequences are exact (ε and ε' are onto). $\exists$ a map $j: \mathbb{Z} \to C_0$ s.t. $\varepsilon \circ j: \mathbb{Z} \to \mathbb{Z}$ is the identity (there is a typo in the book). Now, the top seq splits, $C_0 = \ker \varepsilon \oplus \operatorname{im} j$.

Example This example will help you see why $C_0 = \ker \varepsilon \oplus \operatorname{im} j$. Let $v_0 \in C_0$ be a vertex. Define $j(n) = nv_0 \in C_0$.



C_0 = sequences in here

Let $\alpha = 3v_1 + 2v_2 + 6v_0 \in C_0$.

Let $\alpha' = 3v_0 + 2v_0 + 6v_0 = 11v_0 \in \operatorname{im}(j)$.

Let $\alpha'' = 3v_1 - 3v_0 + 2v_2 - 2v_0 \in \ker \varepsilon$.

Notice $\alpha = \alpha' + \alpha''$.

Now let $j' : \mathbb{Z} \to C_0'$ be $j' = \psi \circ j$. Now

$$C_0' = \ker \varepsilon' \oplus \operatorname{im} j'.$$

And $\psi : \operatorname{im} j \to \operatorname{im} j'$ is an isomorphism because

$$\psi(\cdot) = \varepsilon(id(\varepsilon^{-1}(\cdot)))$$

$$nv_0 \to n \to nv_0$$

$\downarrow$ vertex of k $\hookrightarrow$ but now $x_0 \in X$

Now

$$H_0(\mathcal{C}) \cong \frac{\ker \varepsilon \oplus \operatorname{im} j}{\partial C_1} \cong \frac{\ker \varepsilon}{\partial C_1} \oplus \operatorname{im} j$$

$$H_0(\mathcal{C}') = \frac{\ker \varepsilon' \oplus \operatorname{im} j'}{\partial' C_1'} \cong \frac{\ker \varepsilon'}{\partial' C_1'} \oplus \operatorname{im} j$$

But $\operatorname{im} j \cong \mathbb{Z}$.

$$\tilde{H}_0(\mathcal{C}) \cong \frac{\ker \varepsilon}{\partial C_1}$$

$$\tilde{H}_0(\mathcal{C}') \cong \frac{\ker \varepsilon'}{\partial C_1'}$$

~~Thus, $H_0(\mathcal{C}) \cong H_0($~~

Thus, $\psi_* : H_0(\mathcal{C}) \to H_0(\mathcal{C}')$ is an iso. iff

$\psi_* : \tilde{H}(\mathcal{C}) \to \tilde{H}_0(\mathcal{C}')$ is an iso.

The next lemma is the main result.

Lemma 34.2 $\forall p$, θ induces the isomorphisms:

① $\theta_*: \tilde{H}_p(\mathcal{C}'(K)) \to \tilde{H}_p(|K|)$

② $\theta_*: H_p(\mathcal{C}'(K)) \to H_p(|K|)$

③ $\theta_*: H_p(\mathcal{C}'(K, K_0)) \to H_p(|K|, |K_0|)$.

where K_0 is a subcomplex of K.

Pf I'll only do the finite case.
① ⇒ ② by lemma 34.1. ② ⇒ ③ by the long exact seq of a pair and the 5 lemma:

$$\begin{array}{ccccccccc}
H_p'(K_0) & \to & H_p'(K) & \to & H_p'(K, K_0) & \to & H_{p-1}'(K_0) & \to & H_{p-1}'(K) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\theta_* & & \downarrow\cong & & \downarrow\cong \\
H_p(|K_0|) & \to & H_p(|K|) & \to & H_p(|K|, |K_0|) & \to & H_{p-1}(|K_0|) & \to & H_{p-1}(|K|)
\end{array}$$

⇒ θ_* is an iso.

$n=1$ Suppose K is a finite complex with n simplicies. If $n=1$, K is a vertex and the proof is trivial. See text book.

$n>1$ Suppose ① holds for all complexes with fewer than n simplicies.

Let $\sigma \in K$ be a simplex of max. dimension.
Let $K_0 = K - \{\sigma\}$.
Let Σ be the complex carried by σ (i.e. σ and its faces).
Let $\Sigma_0 = \Sigma - \{\sigma\}$, (i.e. the faces of σ)

Consider
$$\begin{array}{ccc}
\tilde{H}'_p(K) & \xrightarrow{\theta_*} & \tilde{H}_p(|K|) \\
\downarrow i'_* & & \downarrow i_* \\
H'_p(K,\Sigma) & \xrightarrow{\theta_*} & H(|K|,|\sigma|) \\
\uparrow j'_* & & \uparrow j_* \\
H'_p(K_0,\Sigma_0) & \xrightarrow{\theta_*} & H_p(|K_0|,|\partial\sigma|)
\end{array}$$

The i', j', i, j are inclusion maps. θ commutes with inclusion.
(i', j' are simplicial inclusions, i, j are corresponding topological inclusions.)
The θ_* on the bottom is an iso. by the induction hyp.

Since the diagram commutes, if we show i'_*, j'_*, i_*, j_* are isomorphisms, then all the θ_* are iso's.

<u>First i's</u> Consider the long exact seq of the pair $|K|, |\sigma|$:

$$\underset{\overset{\shortparallel}{0}}{H_p(|\sigma|)} \to H_p(|K|) \xrightarrow{i_*} H_p(|K|,|\sigma|) \to \underset{\overset{\shortparallel}{0}}{H_{p-1}(|\sigma|)}.$$

The ends are trivial b/c σ is acyclic. Thus i_* is an iso.
This works for i'_* as well since σ is acyclic in ordered homology.

Now the j's. The inclusion map $j': (K_0, \Sigma_0) \to (K, \Sigma)$, by Thm 9.1 (Excision), induces an iso. of relative simplicial homology gps, which are iso to the ordered homology gps. Thus j'_* is an iso.

Showing the map j_* is an iso is harder. The Excision (31.7) Thm for sing. homology does not apply since ~~$\{|K|-|\sigma|, |\sigma|\}$~~ the ~~an~~ interiors of $\{|K|-|\sigma|, |\sigma|\}$ do not cover $|K|$.

We get around this as follows. Let $\hat{\sigma}$ be the barycenter of σ. Consider

$$\begin{tikzcd}
H_p(|K|, |\sigma|) & \\
 & H_p(|K|-\hat{\sigma}, |\sigma|-\hat{\sigma}) \arrow[ul, "l_*"'] \\
H_p(|K_0|, |\partial\sigma|) \arrow[uu, "j_*"] \arrow[ur, "k_*"'] &
\end{tikzcd}$$

k_* and l_* are induced by inclusion maps k and l. $j = l \circ k \Rightarrow j_* = l_* \circ k_*$. Now l_* is an iso by Excision (31.7). We finish by showing k_* is an iso. as well.

If $\dim \sigma = 0$, this is trivial. $|\sigma| - \hat{\sigma} = \emptyset$, $|\partial\sigma| = \emptyset$ and $k: |K_0| \to |K| - \hat{\sigma}$ is the identity.

~~If $\dim \sigma \geq 1$, then, we claim k is homotopic to the identity. Then Thm 30.8 $\Rightarrow k_*$ is an iso.~~

We claim $k : (|K_0|, |\partial\sigma|) \to (|K| - \hat{\sigma}, |\sigma| - \hat{\sigma})$ is a homotopy equivalence. Then Thm 30.8 $\Rightarrow$ k_* is an iso.

<u>Recall</u> Let $f : (X, A) \to (Y, B)$. If there is a map $g : (Y, B) \to (X, A)$ s.t. $f \circ g$ and $g \circ f$ are homotopic to identity map of pairs, then f is a <u>homotopy equivalence</u>. (pg 174).

Now $|\partial\sigma|$ is a deformation retract of $|\sigma| - \hat{\sigma}$. (Thm 19.6) Let $F : |\sigma| - \hat{\sigma} \times I \to |\sigma| - \hat{\sigma}$ be such a def. retract;

$$F(x, 0) = x \quad \forall x \in |\sigma| - \hat{\sigma}$$
$$F(x, 1) \in |\partial\sigma| \quad \forall x \in |\sigma| - \hat{\sigma}$$
$$F(a, t) = a \quad \forall a \in |\partial\sigma|, t \in I.$$

Extend $F : |K| - \hat{\sigma} \times I \to |K| - \hat{\sigma}$ by defining

$$F(x, t) = x \quad \forall x \in |K_0| = |K| - |\sigma|.$$

The pasting lemma (from 530) shows F is cont.

Let $g : (|K| - \hat{\sigma}, |\sigma| - \hat{\sigma}) \to (|K_0|, |\partial\sigma|)$ be given by $g(x) = F(x, 1)$.

Then $g \circ k = id$ on $(|K_0|, |\partial\sigma|)$.

Now $k \circ g$ is not the identity on $(|K| - \hat{\sigma}, |\sigma| - \hat{\sigma})$ but via F $k \circ g$ is homotopic to the id. $\boxed{\star}$

So, we have a chain map $\theta: C'(K) \to \mathcal{S}(|K|)$. that induces an iso. $\theta_*: H_p'(K) \to H_p(K)$. (ordered homology to singular). Back in §13 we have a ~~map~~ chain map $\phi: C(K) \to C'(K)$ that induced as iso. ~~on~~ $\phi_*: H_p(K) \to H_p'(K)$ (simplicial hom. to ordered homology). Let $\eta = \theta \circ \phi$. Then

$$\underset{\text{simp.}}{\eta_* : H_p(K)} \to \underset{\text{sing.}}{H_p(|K|)} \text{ is an iso.}$$

There are three things left to check that you can read on your own.

<u>Thm</u> (34.3) η_* commutes with ∂_*:

$$\begin{tikzcd}
H_p(K, K_0) \arrow[r, "\partial_*"] \arrow[d, "\eta_*"] & H_{p-1}(K_0) \arrow[d, "\eta_*"] & \text{simp.} \\
H_p(|K|, |K_0|) \arrow[r] & H_{p-1}(|K_0|) & \text{sing}
\end{tikzcd}$$

commutes.

Thm 34.4 η_* commutes with homomorphisms induced by simplicial maps: Let $f:(K,K_0)\to(L,L_0)$ be a simp. map (of pairs). Then is also a cont. map $f:(|K|,|K_0|)\to(|L|,|L_0|)$. Then

$$\begin{tikzcd}
H_p(K,K_0) \arrow[r, "f_*"] \arrow[d, "\eta_*"] & H_p(L,L_0) \arrow[d, "\eta_*"] & \text{simp} \\
H_p(|K|,|K_0|) \arrow[r, "f_*"] & H_p(|L|,|L_0|) & \text{sing}
\end{tikzcd}$$

commutes.

Thm 34.5 η_* commutes with hom's induced by cont. maps in the following sense. Let $h:(|K|,|K_0|)\to(|L|,|L_0|)$ be cont. Let $f:(K',K_0')\to(L,L_0)$ be a simplicial approximation to h. Then

$$\begin{tikzcd}
H_p(K',K_0') \arrow[r, "f_*"] \arrow[d, "g_*"] & H_p(L,L_0) \arrow[dd, "\eta_*"] \\
H_p(K,K_0) \arrow[d, "\eta_*"] & \\
H_p(|K|,|K_0|) \arrow[r, "h_*"] & H_p(|L|,|L_0|)
\end{tikzcd}$$

commutes, where $g:(K_0',K_0')\to(K,K_0)$ is a simp. approx. of the identity.