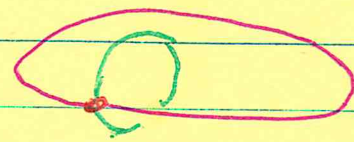
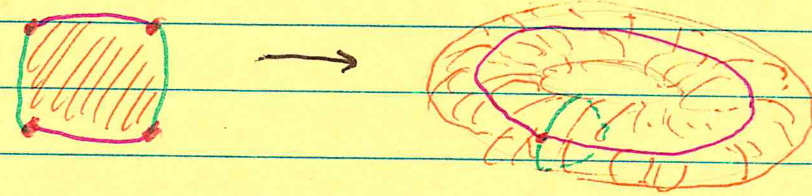
§ 38 CW Complexes

__Example__ Take a pt •, two line segments , and a disk 
Attach the end points of the line segments to the point:



Now attach the bdy of the disk to the structure we just made:



We have constructed the 2-torus as a CW-complex.

__Def__ A space that is homeomorphic to the closed ball B^m is often called a __cell__ of dim m or an __m-cell__. An __open m-cell__ is any space homeo. to $\text{int}(B^m)$. A __0-cell__ is a point.

__Def__ Let $X = \bigcup e_\alpha$ be a disjoint union of open cells and points, 0-cells. Then X is a __CW-complex__ if

① X is Hausdorff,

② $\forall \alpha \ \exists f_\alpha : B^m \to X$ that maps Int B^m homeomorphically onto e_α and carries ~~the~~ bdy of B^m __into__ a finite union of open cells and 0-cells, each of dim $< m$.

③ next pg

③ A subset $A \subset X$ is closed iff $A \cap \bar{e}_\alpha$ is closed in $\bar{e}_\alpha$ for each α.

<u>Def</u> If the f_α's in ② are homeomorphisms on B^m and each $\dot{e} \equiv \bar{e}_\alpha - e_\alpha$ <u>equals</u> the union of finitely many open cells and 0-cells of dim $< m = \dim e_\alpha$, then X is a <u>regular CW-complex</u>.

<u>Rmk</u> There are other equivalent definitions of CW complexes. See <u>The Topology of CW Complexes</u> by Lundell and Weingram. There is a copy in the Math Library. I'll refer to it as LW.

<u>Rmk/Def</u> (Weak Topology) Let X_α be top. sp's; Let $X = \cup X_\alpha$ be a set. The <u>weak topology</u> on X wrt $\{X_\alpha\}$ is given by $U \subset X$ is open iff $U \cap X_\alpha$ is open in X_α for each α. Condition ③ $\Longleftrightarrow$ $X = \cup e_\alpha$ has the weak top wrt $\{\bar{e}_\alpha\}$.

<u>Rmk</u> If a CW complex consists of a finite number of cells then condition ③ is automatic.

Lemma 38.1 Let X be a CW complex.

(a) $f: X \to Y$ is cont. iff $f|_{\bar{e}_\alpha}$ is cont. $\forall \alpha$.

Pf: See Lemma 2.3

(b) $F: X \times I \to Y$ is cont. iff $F|_{\bar{e}_\alpha \times I}$ is cont. $\forall \alpha$.

Pf: See Thm 20.4, and Cor. 20.5.

Fact A CW complex is paracompact (and hence normal).

Pf: See LW Thm 4.2

Q: Let $X_1 = \bigcup e^1_\alpha$ and $X_2 = \bigcup e^2_\beta$ be CW complexes.
Let $X = X_1 \times X_2$ with the prod. top. Let $\tilde{X} = X_1 \times X_2$ with the weak top wrt $\{e^1_\alpha \times e^2_\beta\}$ (each with prod. top.)
When do $X = \tilde{X}$ as top. spaces?

Thm (LW pages 57-58) Let $\eta: X \to \tilde{X}$ be the identity map.
(a) If X_1 ~~and~~ or X_2 is finite or if both are countable then η is a homeo.
(b) If X_1 or X_2 is locally compact, then η is a homeo.

Def Let $X = \bigcup_{\alpha \in \mathcal{A}} e_\alpha$ be a CW complex. A subcomplex is a subset $A \subset X$ s.t. $A = \bigcup_{\alpha \in \mathcal{A}'} e_\alpha$, $\mathcal{A}' \subset \mathcal{A}$ s.t. A is a CW complex.

Def Let X and Y be CW complexes. A map $f: X \to Y$ is cellular if each p-cell of X is taken into the union of open cells of Y of dim at most p.

Thm (LW page 62) Let X and Y be CW complexes. Let $A \subset X$ be a subcomplex. Let $f: A \to Y$ be cellular. Then the adjunction sp. $X \cup_f Y$ is a CW complex. (see pg 210 in your textbook for adjunction spaces.)

Thm (LW page 59) Let $f: X \to X$ be cellular and an equivalence relation. Then the quotient X/f is a CW complex.

Rmk The proof uses normality.

Def Let X be a CW complex. Let $X^p = \bigcup e_\alpha$ s.t. $\dim e_\alpha \leq p$. X^p is call the p-skeleton of X.

Proposition X^p is a CW complex

Pf ① and ② are clear. ③ only becomes an issue for infinite complexes. See textbook, pg 217.

Thm 38.2 Let X be a CW complex of dim p. Then X is homeo. to an adjunction of X^{p-1} and a disjoint union of p-cells. [There is a converse to this as well.]

Pf: See textbook. Compare with Thm 38.3.

Def If X is a top. sp. a filtration of X is a seq of subsets $X_0 \subset X_1 \subset X_2 \subset \cdots$ of subspaces whose union is X. If X and Y have filtrations $X_0 \subset X_1 \subset X_2 \subset \cdots$ and $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ resp., then a continuous map $f: X \to Y$ s.t. $f(X_p) \subset Y_p$, $\forall p$, is said to be filtration-preserving

Ex If X is a CW complex, it skeletons form a filtration: $X^0 \subset X^1 \subset X^2 \subset \cdots$. Cellular maps are filtration-preserving.