§39 Homology Groups of CW Complexes

Singular homology gps are hard to compute directly, but for CW complexes we will develop a tractable method.

Def Let X be a CW complex. Let $D_p(X) = H_p(X^p, X^{p-1})$. Let $\partial : D_p(X) \to D_{p-1}(X)$ be the composite

$$H_p(X^p, X^{p-1}) \xrightarrow{\partial_*} H_{p-1}(X^{p-1}) \xrightarrow{j_*} H_{p-1}(X^{p-1}, X^{p-2})$$

where ∂_* is the connecting homomorphism for the long exact seq of the pair (X^p, X^{p-1}), and j_* is induced by the inclusion map $j : (X^{p-1}, \phi) \to (X^{p-1}, X^{p-2})$.

Fact $\partial \partial = 0$.

Pf $\partial \circ \partial = \partial_* j_* \partial_* j_*$, see below:

$$H_p(X^p, X^{p-1}) \xrightarrow{\partial_*} \underbrace{H_{p-1}(X^{p-1}) \xrightarrow{j_*} H_{p-1}(X^{p-1}, X^{p-2}) \xrightarrow{\partial_*} H_{p-2}(X^{p-2})}_{} \xrightarrow{j_*} H_{p-2}(X^{p-2}, X^{p-3})$$

Look at the middle two hom's. This appears in the long exact seq for the pair (X^{p-1}, X^{p-2}). By exactness $j_* \partial_* = 0$. Hence, so does $\partial \partial$. ∎

Now we explore the chain complex $\{D_p, \partial\} =: \mathcal{D}(X)$.

Our goal is to understan $\mathcal{D}(X)$. We will show each $D_p(X)$ is free abelian and that $H_p(\mathcal{D}(X)) \cong H_p(X)$. It turns out it is easier to compute $H_p(\mathcal{D}(X))$ that it is to complete $H_p(X)$ directly.

We have to cover two lemmas and then Thm 39.3 gives

$$H_i(X^p, X^{p-1}) = 0 \text{ for } i \neq p \text{ and is } = \text{free abelian for } i = p.$$

Then, Thm 39.4 gives us that $H_p(X) \cong H_p(\mathcal{D}(X))$. It proof has three steps.

Step 1 uses the exact seq of a pair to show

$$H_p(X) \cong H_p(X^{p+1}).$$

Step 2 uses the exact seq of a triple to show

$$H_p(X^{p+1}) \cong H_p(X^{p+1}, X^{p-2}).$$

Step 3 uses a braid of exact seq's of triples to show

$$H_p(X^{p+1}, X^{p-2}) \cong H_p(\mathcal{D}(X)).$$

There is a step 4 on naturality.

Hang on to your seats ...

Lemma 39.1 Let e be some open p-cell in a CW complex X. Recall $\bar{e}$ is its closure and $\dot{e} = \bar{e} - e$. Let $f : (B^p, S^{p-1}) \to (\bar{e}, \dot{e})$ where $S^{p-1} = \partial B^p$. Then f induces an isomorphism

$$f_* : H_p(B^p, S^{p-1}) \cong H_p(\bar{e}, \dot{e}).$$

(Recall: $H_p(B^p, S^{p-1}) \cong \mathbb{Z}$.)

Pf Let $\hat{e} = f(0)$. We study the following diagram of cont. maps and the commutative diagram they induce on homology

$$\begin{tikzcd}
(B^p, S^{p-1}) \arrow[r, hook, "i"] \arrow[d, "f"] & (B^p, B^p - 0) \arrow[d, "f'"] & (\mathrm{int}(B^p), \mathrm{int}(B^p) - 0) \arrow[l, hook', "j"'] \arrow[d, "f''"] \\
(\bar{e}, \dot{e}) \arrow[r, hook, "k"] & (\bar{e}, \bar{e} - \hat{e}) & (e, e - \hat{e}) \arrow[l, hook', "l"']
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(B^p, S^{p-1}) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(B^p, B^p - 0) \arrow[d, "f'_*"] & H_n(\mathrm{int}(B^p), \mathrm{int}(B^p) - 0) \arrow[l, "j_*"'] \arrow[d, "f''_*"] \\
H_n(\bar{e}, \dot{e}) \arrow[r, "k_*"] & H_n(\bar{e}, \bar{e} - \hat{e}) & H_n(e, e - \hat{e}) \arrow[l, "l_*"']
\end{tikzcd}$$

i_* and k_* are isomorphisms by Axiom 5 (homotopy) (See Thm 30.8)

j_* and l_* are isomorphisms by Axiom 4 (Excision) (See Thm 31.7)

Now f'' is a homeo, so f''_* is an iso. By commutativity so are f'_* and f_*. ∎

Recall the topological sum of a disjoint union defined on pg 114. It is denoted by ΣE_α for $\{E_\alpha \mid \alpha \in \text{index set}\}$ disjoint.

Lemma 39.2 Let $f : X^{p-1} \underset{\text{disjoint union}}{\sqcup} (\underset{p\text{-cells}}{\Sigma B_\alpha}) \to X^p$ be an adjunction map (gluing) of p-cells onto X^{p-1} to form X^p. Then f induces an isomorphism

$$f_* : H_i(\Sigma B_\alpha, \Sigma \partial B_\alpha) \cong H_i(X^p, X^{p-1}).$$

Pf Similar to 39.1.

Thm 39.3 $H_i(X^p, X^{p-1}) = 0$ for $i \neq p$ and is free abelian for $i = p$.

Pf $H_i(X^p, X^{p-1}) \cong H_i(\Sigma B_\alpha, \Sigma \partial B_\alpha) \cong \bigoplus_\alpha H_i(B_\alpha, \partial B_\alpha) \cong \begin{cases} 0 & i \neq p \\ \oplus \mathbb{Z} & i = p. \end{cases}$

Thm 39.4 $H_p(\mathcal{D}(X)) \cong H_p(X)$.

Pf We will use the fact that if $C \subset X$ is compact, then $\exists p$ s.t. $C \subset X^p$. This is obvious if X is finite dimensional. For the general case see Exercise #1 in §38.

Step 1 We will show $i_*: H_p(X^{p+1}) \to H_p(X)$ is an isomorphism where i_* is induced by inclusion.

Let $i \geq 1$. Consider the exact seq. of the pair (X^{p+i+1}, X^{p+i}).

$$\cdots \to H_{p+1}(X^{p+i+1}, X^{p+i}) \to H_p(X^{p+i}) \to H_p(X^{p+i+1}) \to H_p(X^{p+i+1}, X^{p+i}) \to \cdots$$

Since $i \geq 1$, $H_{p+1}(X^{p+i+1}, X^{p+i})$ and $H_p(X^{p+i+1}, X^{p+i})$ are trivial since there are no singular $p+1$ or p chains once we mod out by $S(X^{p+i})$. It follows that

$$H_p(X^{p+i}) \cong H_p(X^{p+i+1}).$$

Thus, $H_p(X^{p+1}) \cong H_p(X^{p+2}) \cong H_p(X^{p+3}) \cong$ etc.

Now, we have to show $H_p(X^{p+1}) \cong H_p(X)$. If X is

i_* onto — finite dimensional this clear. In general, any element of $H_p(X)$ is represented by a p-chain which has compact support (Axiom 8) and thus is carried by some X^{p+i}. This shows i_* is onto.

i_* 1-to-1 — For injectiveness, let $\beta \in H_p(X^{p+1})$ and suppose $i_*(\beta) = 0 \in H_p(X)$. Since $i_*(\beta)$ is carried by some X^{p+i} it is a member of $H_p(X^{p+i})$. But $H_p(X^{p+1}) \cong H_p(X^{p+i})$ so $i_*(\beta) = 0 \Rightarrow \beta = 0$.

Thus i_* is an iso.

Step 2 We show that $i_* : H_p(X^{p+1}) \to H_p(X^{p+1}, X^{p-2})$ is an iso.
Let $0 \le i \le p-2$. Consider the long exact seq. of the triple (X^{p+1}, X^i, X^{i-1}). [see #2 in §26 and the discussion in ~~your~~ this section in pages 224-2.]

$$\cdots \to \underset{\substack{\shortparallel \\ 0}}{H_p(X^i, X^{i-1})} \to H_p(X^{p+1}, X^{i-1}) \xrightarrow{\cong} H_p(X^{p+1}, X^i) \to \underset{\substack{\shortparallel \\ 0}}{H_{p-1}(X^i, X^{i-1})} \to \cdots$$

Thus the middle groups are isomorphic. Thus,

$$H_p(X^{p+1}, \emptyset) \cong H_p(X^{p+1}, X^1) \cong H_p(X^{p+1}, X^2) \cong \cdots \cong H_p(X^{p+1}, X^{p-2}).$$

Step 3 We will show that $H_p(\mathcal{D}(X)) \cong H_p(X^{p+1}, X^{p-2})$, thus proving the thm. Recall $\mathcal{D}_p(X) = H_p(X^p, X^{p-1})$.

Let $(X, A, B, C) = (X^{p+1}, X^p, X^{p-1}, X^{p-2})$, so $C \subset B \subset A \subset X$.
There are four triples: $(X, A, B), (X, A, C), (X, B, C), (A, B, C)$.
Their corresponding exact seq's can be arranged into an exact braid diagram:

$$\begin{tikzcd}
0 \arrow[d, equal] & & & & 0 \arrow[d, equal] \\
H_p(B,C) \arrow[dr] \arrow[rr, bend left] & & H_p(X,C) \arrow[dr] \arrow[rr, bend left] & & H_p(X,A) \\
& H_p(A,C) \arrow[ur, "l_*"] \arrow[dr, "k_*"] & & H_p(X,B) \arrow[ur] \arrow[dr] & \\
H_{p+1}(X,A) \arrow[ur, "\partial''_*"] \arrow[rr, bend right, "\partial_*"] & & H_p(A,B) \arrow[ur] \arrow[rr, bend right, "\partial'_*"] & & H_{p-1}(B,C) \\
D_{p+1} \arrow[u, equal] & & D_p \arrow[u, equal] & & D_{p-1} \arrow[u, equal]
\end{tikzcd}$$

$H_p(B,C) = H_p(X^{p-1}, X^{p-2}) = 0$ since there are no p-cycles in X^{p-1}.
$H_p(X,A) = H_p(X_{p+1}, X_p) = 0$ since the p-cycles have been modded out.

Since $H_p(B,C) = 0$ the map k_* is 1-to-1.
By exactness k_* carries $H_p(A,C)$ onto $\ker \partial_*'$.
Thus $H_p(A,C) \cong \ker \partial_*'$.

Since $H_p(X,A) = 0$, the map l_* is onto.
Thus $l_* \circ k_*^{-1}$ takes $\ker \partial_*'$ onto $H_p(X,C)$.

The ker of l_* is $\operatorname{im} \partial_*''$ by exactness.

The bottom row of the braid is $D_{p+1} \xrightarrow{\partial_*} D_p \xrightarrow{\partial_*'} D_{p-1}$.
It is not exact, but $\mathcal{D}(X)$ is a chain complex.
↑ original X not X^{p+1}

Thus, $\operatorname{im} \partial_* \subset \ker \partial_*'$. Notice $\frac{\ker \partial_*'}{\operatorname{im} \partial_*} = H_p(\mathcal{D}(X))$.

If we replace $H_p(A,B)$ by $\ker \partial_*' \cong H(A,C)$, we can regard ∂_*'' and ∂_* as the same.

Thus, $\ker l_* \circ k_*^{-1}$ is the same as l_* which $= \operatorname{im} \partial_*''$ by exactness. Now we have

$$H_p(\mathcal{D}(X)) = \frac{\ker \partial_*'}{\operatorname{im} \partial_*} \cong H_p(X,C) \cong H_p(X^{p+1}, X^{p-2}).$$

Combining with Steps 1 and 2 we have the result.

Step 4 The book goes on to show that the isomorphism

$$H_p(\mathcal{D}(X)) \cong H_p(X)$$

is "natural," meaning if $f: X \to Y$ is ~~cont.~~ cellular and takes X^p into Y^p then

$$\begin{tikzcd}
H_p(\mathcal{D}(X)) \arrow[r, phantom, "\cong"] \arrow[d, "f_*"] & H_p(X) \arrow[d, "f_*"] \\
H_p(\mathcal{D}(Y)) \arrow[r, phantom, "\cong"] & H_p(Y)
\end{tikzcd}$$

commutes. See textbook.