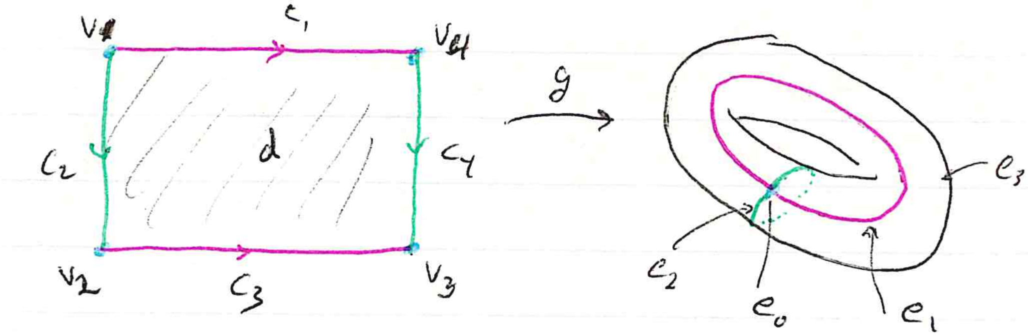
Ex <u>The Torus</u>. Let $g : R \to T$ where R is the rectangle below and T is the 2-torus.



T is the union of the 0-cell e_0, two open 1-cells e_1, e_2, and an open 2-cell e_3. Thus,

$$D_2(T) \cong \mathbb{Z} \quad D_1(T) \cong \mathbb{Z}^2 \quad D_0(T) \cong \mathbb{Z}.$$

R is the union of 4 0-cells, v_1, v_2, v_3, v_4, 4 1-cells, c_1, c_2, c_3, c_4 and 1 2-cell d.

By Lemma 39.1 the map $g\,(\bar{d} = R,\ \partial R = c_1 \cup c_2 \cup c_3 \cup c_4) \to (\bar{e}_3, \dot{e}_3)$ induces an isomorphism:

$$H_i(R, \partial R) \cong H_i(\bar{e}_3, \dot{e}_3)$$

Let $\gamma = g_\#(\bar{d})$. γ is a generator of $H_2(\bar{e}_3, \dot{e}_3) = D_2(T)$.

Let $w_1 = g_\#(\bar{c}_1) = g_\#(\bar{c}_3)$ and $z_1 = g_\#(\bar{c}_2) = g_\#(\bar{c}_4)$. Then, using 39.1 w_1 is a generator of $H_1(\bar{e}_1, \dot{e}_1)$ and z_1 is a generator of $H_1(\bar{e}_2, \dot{e}_2)$. Together they ~~gen~~ form a basis of $D_1(T)$ by Lemma 39.2.

Now we study the bdy maps $D_2(\mathbb{T}) \xrightarrow{\partial} D_1(\mathbb{T}) \xrightarrow{\partial} D_0(T)$.

$$\partial w_1 = \partial g_\# c_1 = g_\# \partial c = g_\#(v_4 - v_1) = e_0 - e_0 = 0.$$

$$\partial z_1 = 0 \quad \text{likewise.}$$

$$H_0(\mathcal{D}(T)) = \frac{D_0(T)}{0} = D_0(T) \cong \mathbb{Z}.$$

$$\partial \gamma = g_\#(\partial d) = g_\#(-c_1 + c_2 + c_3 - c_4) = 0$$

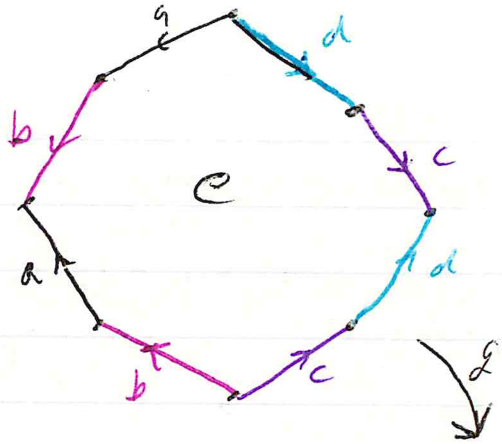
Thus, $H_1(\mathcal{D}(T)) = \dfrac{D_1(T)}{0} \cong \mathbb{Z}^2$.

And $H_2(\mathcal{D}(T)) = \dfrac{D_2(T)}{0} \cong \mathbb{Z}$.

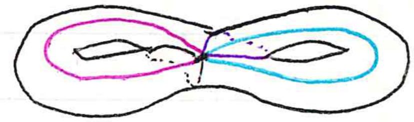
Thus, $H_2(T) \cong \mathbb{Z}$, $H_1(T) \cong \mathbb{Z}^2$, $H_0(T) \cong \mathbb{Z}$.

I'll do two more examples, but without writing out all the details.

Ex Let $X = T^2 \# T^2$



$D_2 \cong \mathbb{Z}$
$D_1 \cong \mathbb{Z}^4$
$D_0 \cong \mathbb{Z}$



$$0 \to D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \to 0$$

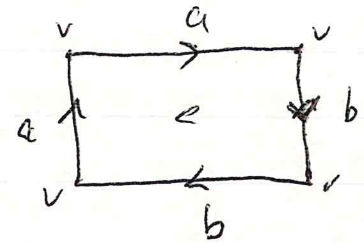
$$\partial g_{\#} e = a + b - a - b + c + d - c - d = 0.$$

$$\partial g_{\#} a = v - v = 0 \quad \text{etc.}$$

$\ker \partial_2 = D_2 \qquad \operatorname{im} \partial_2 = 0$
$\ker \partial_1 = D_1 \qquad \operatorname{im} \partial_1 = 0$

$H_2(X) \cong \mathbb{Z}, \quad H_1(X) \cong \mathbb{Z}^4, \quad H_0(X) \cong \mathbb{Z}.$

Ex $K = P^2 \# P^2$



$D_2 \cong \mathbb{Z}, \quad D_1 \cong \mathbb{Z}^2 \quad D_0 \cong \mathbb{Z}.$

$H_2 = \dfrac{\ker \partial_2}{0}$

$\partial_2 e = a + a + b + b = 2(a+b)$

$\partial_2(ne) = 0$ iff $n = 0$.

Thus, $\ker \partial_2 = 0$.

Thus, $H_2 = 0$.

$H_1 = \dfrac{\ker \partial_1}{\mathrm{im}\, \partial_2}$

$\partial_1(a) = v - v = 0$

$\partial_1(b) = v - v = 0$

$\ker \partial_1 = D_1 \cong \mathbb{Z}^2$.

$\mathrm{im}\, \partial_2 = \langle 2(a+b) \rangle$.

$H_1 \cong \dfrac{\mathbb{Z}^2}{\cancel{\mathrm{im}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \mathbb{Z}^2} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (Smith normal form).

$H_0 \cong D_0 \cong \mathbb{Z}$.