§39, #2 Let $A \subset U \subset X$, where $A$ is closed, $U$ is open, and $A$ is a deformation retract of $U$. Define an equivalence relation on $X$ by $x \sim y$ if $x = y$ or, $x$ and $y$ are both in $A$. Thus $[x] = \{x\}$ if $x \notin A$ and $[x] = A$ if $x \in A$. Let $X/A = \{[x] \mid x \in X\}$. Define $q : X \to X/A$ by $q(x) = [x]$. Give $X/A$ the quotient topology. When in $X/A$ let $a$ denote $A$.

~~[illegible]~~ Prove that $H_p(X, A) \cong H_p(X/A)$.

Comparison to Lemma 39.1

| | | | |
|---|---|---|---|
| $X \longleftrightarrow B^p$ | | $X/A \longleftrightarrow \bar{e}_\alpha$ | |
| $U \longleftrightarrow B^p - 0$ | | $a \longleftrightarrow \dot{e}_\alpha$ | |
| $A \longleftrightarrow S^{p-1} = \partial B^p$ | | $U/A \longleftrightarrow \bar{e}_\alpha - \hat{e}_\alpha$ | |
| $X - A \longleftrightarrow \operatorname{Int} B^p$ | | $X/A - a \longleftrightarrow e_\alpha$ | |
| $U - A \longleftrightarrow \operatorname{Int} B^p - 0$ | | $U/A - a \longleftrightarrow e_\alpha - \hat{e}_\alpha$ | |

In one sense this problem is more general than 39.1; $(X, A)$ is more general than $(B^p, S^{p-1})$. But, in another sense it is more specific; we are collapsing $A$ to a point whereas in 39.1 $S^{p-1}$ can be mapped to many different subsets, $\dot{e}_\alpha$ of the $p-1$-skeleton of the CW complex.

We use $q$ to also denote the mapping between pairs

$$q:(X,A) \to \left(\tfrac{X}{A}, a\right)$$

and use $q'$ and $q''$ as in this diagram:

$$\begin{tikzcd}
(X,A) \arrow[r, hook, "\alpha"] \arrow[d, "q"] & (X,U) \arrow[d, "q'"] & (X-A, U-A) \arrow[l, hook', "\gamma"'] \arrow[d, "q''"] \\
\left(\tfrac{X}{A}, a\right) \arrow[r, hook, "\beta"] & \left(\tfrac{X}{A}, \tfrac{U}{A}\right) & \left(\tfrac{X}{A}-a, \tfrac{U}{A}-a\right) \arrow[l, "\delta"']
\end{tikzcd}$$

$\alpha, \beta, \gamma, \delta$ are inclusions. Since $A$ is a deformation retract of $U$ (just as $\partial B^p$ was a def ret. of $B^p - 0$) Thm 30.8 gives $H_p(X,A) \cong H_p(X,U)$. But we can factor the deformation through $q$ and get $H_p(\tfrac{X}{A}, a) \cong H_p(\tfrac{X}{A}, \tfrac{U}{A})$.

The $\gamma$ and $\delta$ are ~~[illegible]~~ excisions and ~~so~~ so the corresponding homology groups are iso. by Thm 31.7 (Note: the roles of $A$ and $U$ are reversed: $\bar{A} = A \subset U = \operatorname{int} U$.)

So, now we have,

$$\begin{tikzcd}
H_p(X,A) \arrow[r, "\cong"] \arrow[d, "q_*"] & H_p(X,U) \arrow[d, "q'_*"] & H_p(X-A, U-A) \arrow[l, "\cong"'] \arrow[d, "q''_*"] \\
H(\tfrac{X}{A}, a) \arrow[r, "\cong"] & H_p\left(\tfrac{X}{A}, \tfrac{U}{A}\right) & H_p\left(\tfrac{X}{A}-a, \tfrac{U}{A}-a\right) \arrow[l, "\cong"']
\end{tikzcd}$$

and this commutes

Now, $q''$ is one-to-one, onto, that is, it is a homeo. Thus $q''_*$ is an isomorphism. By communitivity we have $q_*$ is an iso. Thus

$$H_p(X,A) \cong H_p(X/A, a) \cong \tilde{H}_p(X/A)$$

see #3 in §9.