§40

I. Homology groups of finite dim. real projective spaces.
The text does the infinite dim case and does complex projective spaces. Read about those on your own if you like.

First a quick alg. calculation.

<u>Ex 1</u> Find the homology groups for the seq (not exact) of groups (all $\mathbb{Z}$) and homomorphisms:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \longrightarrow 0$$

$$p = 4 \qquad 3 \qquad 2 \qquad 1 \qquad 0$$

Since im of each $\subset$ the ker of the next, homology groups are defined.

$H_0 = \mathbb{Z}/0 = \mathbb{Z}$

$H_1 = \mathbb{Z}/2\mathbb{Z}$

$H_2 = 0/0 = 0$

$H_3 = \mathbb{Z}/2\mathbb{Z}$

$H_4 = 0/0 = 0$

<u>Ex 2</u> Consider: $0 \to \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \to 0$

Then

$H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/2\mathbb{Z}$, $H_2 = 0$, $H_3 = \mathbb{Z}/2\mathbb{Z}$, $H_4 = 0$, $H_5 = \mathbb{Z}$.

<u>Generalize.</u>

Def Let $a : S^n \to S^n$ be the *antipodal map*.

$$a(x_1, x_2 \ldots x_n, x_{n+1}) = (-x_1, -x_2, \ldots -x_n, -x_{n+1}).$$

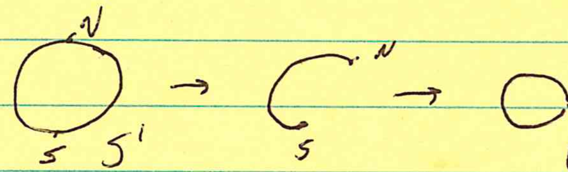
Then $a_* : H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$.

This number is the *degree* of a (see Thm 21.3). $n \geq 1$

For $n \geq 0$, let $P^n = S^n/a$. P^n is called the n-dim real projective space.

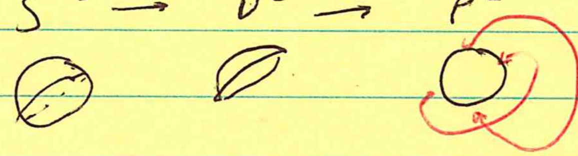
Ex P^0 = a point.

P^1:



$P^1 \approx S^1$.

P^2: $S^2 \to D^2 \to P^2$



P^3: $S^3 \to B^3 \to P^3$

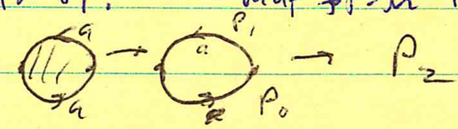
think a 3-ball with opposite pts of bdy S^2 identified.

etc.

CW We can make P^n into a CW complex with one cell in each dimension, $0, 1, 2, \ldots n$.

For P^0 this is obvious since it is a 0-cell. P^1 is also clear as we can take a 1-cell and attach each end pt to the 0-cell of P^0.

For P^2 we can attach a two-cell to P_1. The map from the $\partial B^2 \to P_1$ is 2-to-1.



Etc. The textbook give a detailed proof using induction; see Thm 40.1.

Thm 40.6

$$H_i(P^{2n+1}) \cong \begin{cases} \mathbb{Z} & i = 2n+1,\ i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1, 3, \ldots, 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(P^{2n}) \cong \begin{cases} 0 & i = 2n \\ \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1, 3, \ldots, 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Ex $H_0(P^3) \cong \mathbb{Z}$, $H_1(P^3) \cong \mathbb{Z}/2\mathbb{Z}$, $H_2(P^3) = 0$, $H_3(P^3) \cong \mathbb{Z}$.

$H_0(P^4) \cong \mathbb{Z}$, $H_1(P^4) \cong \mathbb{Z}/2\mathbb{Z}$, $H_2(P^4) = 0$, $H_3(P^4) \cong \mathbb{Z}/2\mathbb{Z}$, $H_4(P^4) = 0$.

Pf The proof follows from Thm 40.5 + Lemma 40.4, and the calculations we did in the examples.

We want to find the homology groups of the chain complex $\mathcal{D}(P^n)$. We now know each $D_k(P^n) \cong \mathbb{Z}$, $k=0,\dots,n$. $D_k(P^n) = H_k(P^k, P^{k-1})$ and so we need to understand the boundary maps.

<u>Thm 40.5</u> $\partial : H_{n+1}(P^{n+1}, P^n) \to H_n(P^n, P^{n-1}) = \begin{cases} 0 & n \text{ even} \\ \times 2 & n \text{ odd.} \end{cases}$

<u>Pf</u> The restriction of the quotient map $p : S^{n+1} \to P^{n+1}$ to the upper hemisphere $p : E_+^{n+1} \to P^{n+1}$ is a characterist map: it is 1-to-1 on $\operatorname{int}(E_+^{n+1})$ and takes $\partial E_+^{n+1} = S^n =$ equator of S^{n+1} into P^n.

Consider the commutative diagram

$$\begin{tikzcd} H_{n+1}(E_+^{n+1}, S^n) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "p_*", "\cong"'] & H_n(S^n) \arrow[d, "p_*"] & \\ H_{n+1}(P^{n+1}, P^n) \arrow[r, "\partial_*"] & H_n(P^n) \arrow[r, "j_*"] & H_n(P^n, P^{n-1}) \end{tikzcd}$$

$\cong$ is an iso by the long exact seq. for (E_+^{n+1}, S^n).

$\cong$ is an iso by Thm 39.3.

~~Thus, since $(j \circ p)_* = \{$~~

The next lemma shows $(j \circ p)_* = \begin{cases} \times 0 & n \text{ even} \\ \times 2 & n \text{ odd.} \end{cases}$

Thus $(j \circ \partial)_*$ must do the same. But this is the bdy map $\partial : H_{n+1}(P^{n+1}, P^n) \to H_n(P^n, P^{n-1})$ by definition (pg 222).

Lemma 40.4 Recall $p: S^n \to P^n$ is the q-map: $S^n \to S^n/a$ $(a^2=1)$ and $j: (P^n, \phi) \to (P^n, P^{n-1})$ is inclusion. Then $(j \circ p)_* = \begin{cases} \times 0 & n \text{ even} \\ \times 2 & n \text{ odd.} \end{cases}$

$$H_n(S^n) \xrightarrow{p_*} H_n(P^n) \xrightarrow{j_*} H_n(P^n, P^{n-1}).$$

Outline of Pf. Let c be a relative cycle s.t. $[c]$ generates $H_n(E^n_+, S^{n-1})$.

Then $[p_\# c]$ generates $H_n(P^n, P^{n-1})$ by Lemma 39.1 (pg 222).
Let $\gamma = c + (-1)^{n-1} a_\# c \in$ n-chains of S^n. Claim: $\partial\gamma = 0$.

Pf $\partial\gamma = \partial c + (-1)^{n-1} \partial a_\# c = \partial c + (-1)^{n-1} a_\# \partial c$
$= \partial c + (-1)^{n-1} (-1)^n \partial c = (1-1)\partial c = 0.$

Claim: $[\gamma]$ is a generator of $H_n(S^n)$. pf: It cannot be a multiple of another cycle since its restriction to E^n_+ is a generator for $H_n(E^n_+, S^{n-1})$.

$p_\#(\gamma) = p_\#(c + (-1)^{n-1} a_\# c) = p_\# c + (-1)^{n-1} p_\# a_\# c$ (but $p \circ a = p$)

$= (1 + (-1)^{n-1}) p_\# c = \begin{cases} 0 & \text{if } n \text{ even} \\ 2p_\# c & \text{if } n \text{ odd.} \end{cases}$

j_* must take 0 to 0 and a generator to a generator since it comes from inclusion. $\square$