Ch. 5 Cohomology

§41 The Hom Functor

Def Let A and B be abelian groups.
Let $\mathrm{Hom}(A,B)$ = all homomorphisms from A to B.
It becomes an abelian gp itself using the addition operatation in B: (If $f, g \in \mathrm{Hom}(A,B)$, define $(f \oplus g)(a) = f(a) + g(a)$.)

Ex $\mathrm{Hom}(\mathbb{Z}, G) \cong G$: $f: \mathbb{Z} \to G$ is determined by $f(1) \in G$.
$\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) = 0$: Let $f: \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}$.
If $f(1) = n \neq 0$, then $f(1+1+1) = 3n \neq 0$, but $1+1+1 = 0$ in $\mathbb{Z}/3\mathbb{Z}$. Thus $f(1) = 0$.
$\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$: Let $f: \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$. Where could $f(1)$ go? It can only go to 0, 2 or 4.

Q: Later we will find a general rule for $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$.

Def Let G, A and B be abelian gps. Let $f \in \mathrm{Hom}(A,B)$. Define

$$\tilde{f} : \mathrm{Hom}(B,G) \to H(A,G)$$

by

$$\tilde{f}(\phi) = \phi \circ f.$$

$\tilde{f}$ is the <u>dual hom.</u> of f (wrt G)

Notice: $\tilde{f} \in \mathrm{Hom}\big(\mathrm{Hom}(B,G), \mathrm{Hom}(A,G)\big)$.

<u>Reversing arrows:</u>

$$A \xrightarrow{f} B$$

$$\mathrm{Hom}(A,G) \xleftarrow{\tilde{f}} \mathrm{Hom}(B,G)$$

For a given G the maping $(A,f) \to (\mathrm{Hom}(A,G), \tilde{f})$ determines a <u>contravariant functor</u>. See §28, pg 158.

41.1 <u>Fun Facts</u>

(a) If f is an iso., so is $\tilde{f}$.

(b) If f is the zero hom., so if $\tilde{f}$.

(c) If f is onto, then $\tilde{f}$ is one-to-one.

We will prove (c).

Proof of (c) Suppose $f: A \to B$ is onto. Let $\phi, \psi \in Hom(B, G)$.
Suppose $\tilde{f}(\phi) = \tilde{f}(\psi)$ in $Hom(A, G)$.
That is, $\forall a \in A$ we have

$$\tilde{f}(\phi)(a) = \tilde{f}(\psi)(a)$$

or $$\phi \circ f(a) = \psi \circ f(a).$$

Let $b \in B$. $\exists a' \in A$ s.t. $f(a') = b$, since f is onto.
Then

$$\tilde{f}(\phi)(a') = \phi \circ f(a') = \phi(b).$$

and $$\tilde{f}(\psi)(a') = \psi \circ f(a') = \psi(b).$$

Since $\tilde{f}(\phi)(a') = \tilde{f}(\psi)(a')$ we have $\phi(b) = \psi(b)$.
This holds $\forall b \in B$, thus $\phi = \psi$ and $\tilde{f}$ is one-to-one.

Another way to express this is to say

if $A \xrightarrow{f} B \to 0$ is exact

then $Hom(A, G) \xleftarrow{\tilde{f}} Hom(B, G) \longleftarrow 0$ is exact.

We can extend this idea.

41.2 If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then the dual seq

$$\mathrm{Hom}(A,G) \xleftarrow{\tilde{f}} \mathrm{Hom}(B,G) \xleftarrow{\tilde{g}} \mathrm{Hom}(C,G) \leftarrow 0$$

is exact. Also, if f is 1-to-1 and the first seq splits, the $\tilde{f}$ is onto and the dual seq splits.

Pf See textbook. [Review pf of 23.1 (2), pg 131-2 first.]

Note It is not true in general that $0 \to A \to B \to C \to 0$ exact implies the dual seq is exact. See Example on pg 248 after the proof of 41.2. See Exercise 4 for when the dual seq is exact (G is a division alg.). See §52 for how the Ext functor "measures" ~~the degree~~ how far from being exact the dual seq. is.

Def Given $\alpha \in \mathrm{Hom}(A, A')$, $\beta \in \mathrm{Hom}(G', G)$ define

$$\mathrm{Hom}(\alpha,\beta) : \mathrm{Hom}(A', G') \to \mathrm{Hom}(A, G) \text{ by}$$

Sending $(\phi' : A' \to G')$ to $(\beta \circ \phi' \circ \alpha) : A \to G$.

Now Hom can be considered as a functor not just in the first variable but in both! see textbook.

Thm 41.3 ⓐ $\mathrm{Hom}\left(\hat{\bigoplus}_{i=1} A_i, G\right) \cong \hat{\bigoplus}_{i=1} \mathrm{Hom}(A_i, G)$

ⓑ $\mathrm{Hom}\left(A, \hat{\bigoplus}_{i=1} G\right) \cong \hat{\bigoplus}_{i=1} \mathrm{Hom}(A, G_i)$

ⓒ $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \cong \ker\left(G \xrightarrow{\times m} G\right)$.

Note: Book's version of this thm is more general.

Pf of ⓒ The sequence $(m>0)$ $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{\text{mod } m} \mathbb{Z}/m\mathbb{Z} \to 0$ is exact. Thus,

$$\mathrm{Hom}(\mathbb{Z}, G) \xleftarrow{\widetilde{\times m}} \mathrm{Hom}(\mathbb{Z}, G) \xleftarrow{\widetilde{\text{mod } m}} \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \leftarrow 0$$

is exact. Since $\text{mod } m : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is onto, $\widetilde{\text{mod } m}$ is one-to-one. Thus,

$$\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \cong \mathrm{im}(\widetilde{\text{mod } m}) \cong \ker(\widetilde{\times m}).$$

We claim $\ker(\widetilde{\times m}) \cong \ker\left(G \xrightarrow{\times m} G\right)$. Consider

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}, G) \arrow[d, phantom, "\rho\,\|\wr"] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[l, "\widetilde{\times m}"'] \arrow[d, phantom, "\rho\,\|\wr"] \\
G & G \arrow[l, "\times m"']
\end{tikzcd}$$

$g \in G$

Let $g \in G$ (lower right corner).
Let $\phi_g \in \mathrm{Hom}(\mathbb{Z}, G)$ be determined by $\phi_g(1) = g$.

Then define $p(\phi_g) = g$. Then p is an iso. Now

$$\widetilde{\times m}(\phi_g)(1) = \phi_g(m \times 1) = m\,\phi_g(1) = mg.$$ Thus,

the diagram commutes. Thus $\ker(\widetilde{\times m}) \cong \ker(\times m)$. $\square$

Lemma 41.4 Let m and n be positive integers. Let $d = \gcd(m,n)$. Then the seq

$$0 \to \frac{\mathbb{Z}}{d\mathbb{Z}} \xrightarrow{\times \frac{n}{d}} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\times m} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\text{mod } d} \frac{\mathbb{Z}}{d\mathbb{Z}} \to 0$$

is exact.

Pf Exercise 3. See next page for an example

Fun Fact $\operatorname{Hom}\left(\frac{\mathbb{Z}}{m\mathbb{Z}}, \frac{\mathbb{Z}}{n\mathbb{Z}}\right) \cong \frac{\mathbb{Z}}{d\mathbb{Z}}$.

Pf $\operatorname{Hom}\left(\frac{\mathbb{Z}}{m\mathbb{Z}}, \frac{\mathbb{Z}}{n\mathbb{Z}}\right) \underset{41.3c}{\cong} \ker\left(\frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\times m} \frac{\mathbb{Z}}{n\mathbb{Z}}\right) \underset{41.4}{\cong} \frac{\mathbb{Z}}{d\mathbb{Z}}$.

Ex $\operatorname{Hom}\left(\mathbb{Z} \oplus \frac{\mathbb{Z}}{15\mathbb{Z}}, \frac{\mathbb{Z}}{12\mathbb{Z}} \oplus \frac{\mathbb{Z}}{10\mathbb{Z}}\right) \cong$

$$\operatorname{Hom}\left(\mathbb{Z}, \frac{\mathbb{Z}}{12\mathbb{Z}}\right) \oplus \operatorname{Hom}\left(\mathbb{Z}, \frac{\mathbb{Z}}{10\mathbb{Z}}\right) \oplus \operatorname{Hom}\left(\frac{\mathbb{Z}}{15\mathbb{Z}}, \frac{\mathbb{Z}}{12\mathbb{Z}}\right) \oplus \operatorname{Hom}\left(\frac{\mathbb{Z}}{15\mathbb{Z}}, \frac{\mathbb{Z}}{10\mathbb{Z}}\right)$$

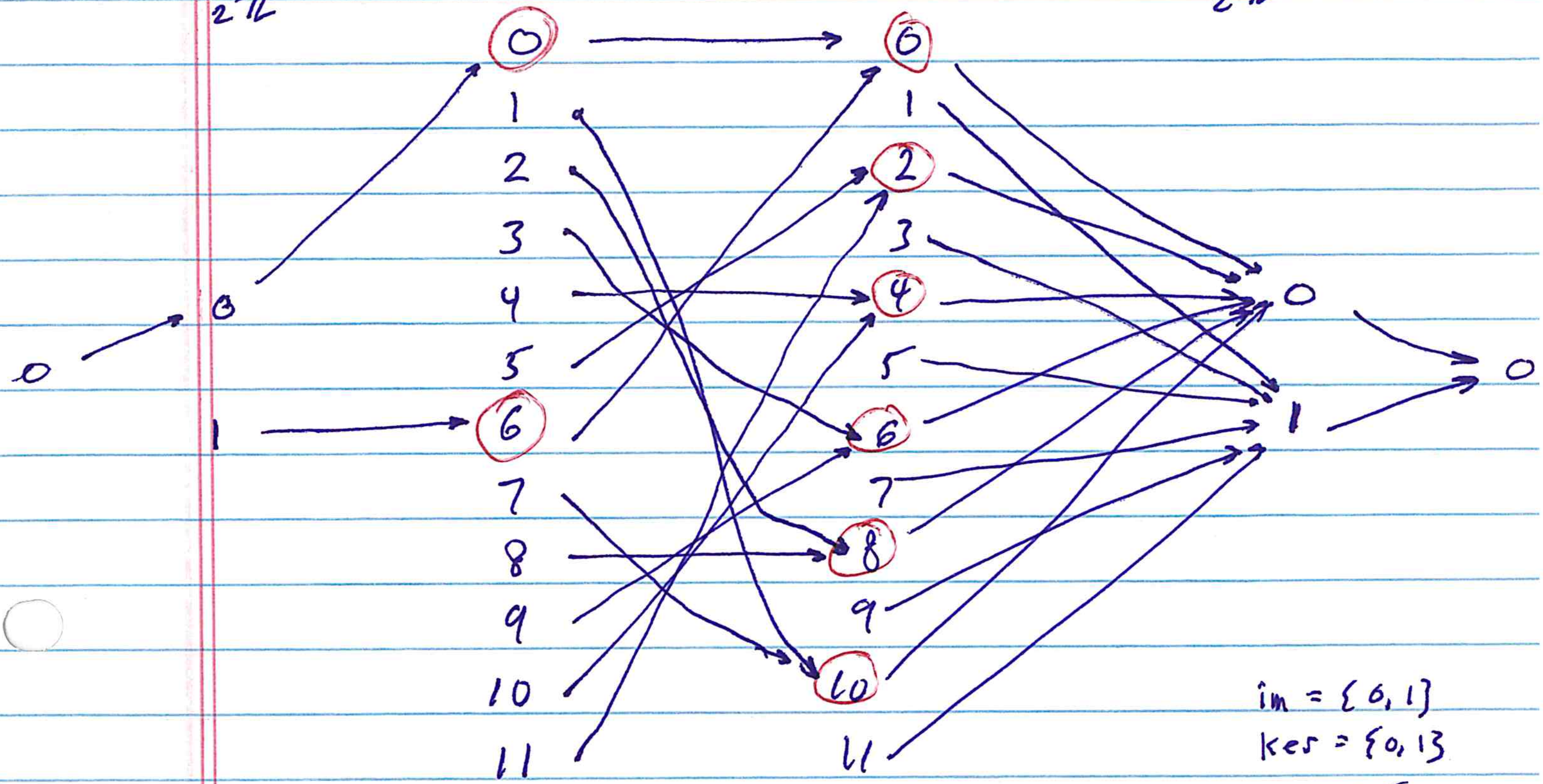
$$\cong \frac{\mathbb{Z}}{12} \oplus \frac{\mathbb{Z}}{10} \oplus \frac{\mathbb{Z}}{3} \oplus \frac{\mathbb{Z}}{5} \cong \frac{\mathbb{Z}}{30} \oplus \frac{\mathbb{Z}}{60}$$

(Smith normal form, check this)

Example for Lemma 41.4.

Let $n=12$, $m=10$. Then $d = \gcd(12,10) = 2$. $\frac{n}{d} = 6$.

$$0 \xrightarrow{\text{inclusion}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 6} \mathbb{Z}/12\mathbb{Z} \xrightarrow{\times 10} \mathbb{Z}/12\mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 0} 0$$



$\text{im} = 0$ $\ker = 0$ ✓

$\text{im} = \{0,6\}$
$\ker = \{0,6\}$ ✓

$\text{im} = \{0,2,4,6,8,10\}$
$\ker = \{0,2,4,6,8,10\}$ ✓

$\text{im} = \{0,1\}$
$\ker = \{0,1\}$ ✓

It is exact!

(~~Example on~~ back side)

Exercise 3 Optional Reading

Lemma 4.1.4 Let m and n be positive integers; let $d = \gcd(m,n)$. Then

$$0 \longrightarrow Z_d \xrightarrow{\frac{n}{d}} Z_n \xrightarrow{m} Z_n \xrightarrow{\text{mod } d} Z_d \longrightarrow 0$$

is exact.

Proof ① The $\ker(\frac{n}{d})$ is clearly 0.

② ⓐ Let $k \in im(\frac{n}{d}) \subset Z_n$. Then $k = k' \frac{n}{d}$.

Thus, $mk = mk'\frac{n}{d} = \frac{m}{d}k'n = 0 \bmod n$.

Hence, $im(\frac{n}{d}) \subset \ker(m)$.

ⓑ Let $k \in \ker(m)$, i.e., $mk = qn$. Let $m' = \frac{m}{d}$; $n' = \frac{n}{d}$.

Now, $m'dk = qn'd \Rightarrow m'k = qn'$. Since m' and n' have no common factors it must be that $m' | q$.

Thus, $k = \left(\frac{q}{m'}\right)n' \Rightarrow k \in im(\frac{n}{d})$. $\Rightarrow \ker(m) \subset im(\frac{n}{d})$.

This proves $\ker(m) = im(\frac{n}{d})$.

③ ⓐ Let $k \in im(m)$. Then $k = k'm + qn = k'm'd + qn'd = (k'm' + qn')d = 0 \bmod d$. Thus, $im(m) \subset \ker(\bmod d)$.

ⓑ Let $k \in \ker(\bmod d)$. Thus, $k = k'd$.

Now, $d = pm + qn$ for some p and q.

Thus, $k = k'(pm + qn) = p'm + q'n$. We require $p' \in Z_n = \{0, \dots, n-1\}$. If it is not then $p' = p'' + q''n$ where $p'' \in Z_n$. Then $k = p''m + (q''m + q')n$.

Thus $k \in im(m)$. Hence $\ker(\bmod d) \subset im(m)$.

This proves the lemma. $\blacksquare$