§42 Simplicial Cohomology

Def Let K be a simplicial complex. The group of p-cochains of K is

$$C^p(K) = \operatorname{Hom}(C_p(K), \mathbb{Z}).$$

The coboundary map δ is the dual of ∂. That is $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ induces

$$\operatorname{Hom}(C_{p+1}(K), \mathbb{Z}) \xleftarrow{\delta = \tilde{\partial}} \operatorname{Hom}(C_p(K), \mathbb{Z})$$

or $$\delta_{p+1} : C^p(K) \to C^{p+1}(K).$$

We define...

$$Z^p(K) = \ker(\delta_{p+1} : C^p(K) \to C^{p+1}(K)) \quad \text{p-cocycles}$$

$$B^p(K) = \operatorname{im}(\delta_p : C^{p-1}(K) \to C^p(K) \quad \text{p-coboundaries}$$

$$H^p(K) = Z_p(K) / B_p(K) \quad , \quad p^{th} \text{ cohomology group of } K.$$

Bracket Notation

Let $a^p \in C^p(K)$, $b_p \in C_p(K)$. $a^p : C_p(K) \to \mathbb{Z}$.
So, $a^p(b_p) \in \mathbb{Z}$. We sometime write $a^p(b_p)$ as

$$\langle a^p, b_p \rangle.$$

Then, $\langle \delta a^p, b_{p+1} \rangle = \delta a^p(b_{p+1}) = a^p(\partial b_{p+1}) = \langle a^p, \partial b_{p+1} \rangle$.

Ex $K = v_1 \xrightarrow{e_1} v_2$. $\partial e_1 = v_2 - v_1$.

$C_0 = \langle v_1, v_2 \rangle$, $C_1 \langle e_1 \rangle$

Define $v_i^* : C_0(K) \to \mathbb{Z}$ by $v_i^*(v_j) = \delta_{ij}$.
For example $v_1^*(7v_1 - 3v_2) = 7$.
Then $C^0(K)$ is generated by $\{v_1^*, v_2^*\}$.

Define $e_i^* : C_1(K) \to \mathbb{Z}$ be $e_i^*(e_j) = \delta_{ij}$.
Then $C^1(K) = \langle e_1^* \rangle$.

Consider $0 \longrightarrow C^0(K) \xrightarrow{\delta} C^1(K) \to 0$.

What is $\delta(v_1^*)$? It maps (eats) edges to $\mathbb{Z}$.

$\delta(v_1^*)(e_1) = v_1^*(\partial e_1) = v_1^*(v_2 - v_1) = -1$.

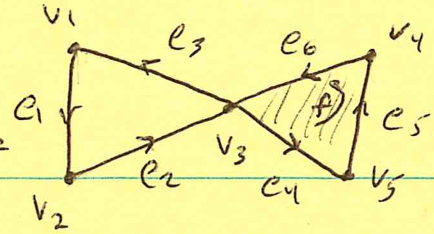
But this is exactly what $-e_1^* : C_1(K) \to \mathbb{Z}$ does:
$-e_1^*(e_1) = -\delta_{11} = -1$.

Thus, $\delta(v_1^*) = -e_1^*$.

Now $\delta(v_2^*)(e_1) = v_2^*(v_2 - v_1) = 1$ so $\delta(v_2^*) = e_1^*$.

The image of δ is all of $C^1(K)$. The kernel is generated by $v_1^* + v_2^*$. Thus,

$$H^0(K) = \frac{\langle v_1^* + v_2^* \rangle}{0} \cong \mathbb{Z} \text{ and } H^1(K) = \frac{C^1(K)}{C^1(K)} = 0.$$

Ex $K =$ 

$C_0(K) = \langle v_1, \dots, v_5 \rangle$
$C_1(K) = \langle e_1 \dots e_6 \rangle$
$C_2(K) = \langle f \rangle$.

The cochain group are generated by ...

$(p=0)$ v_i^* where $v_i^*(v_j) = \delta_{ij}$, $i = 1 \dots 5$

$(p=1)$ e_i^* where $e_i^*(e_j) = \delta_{ij}$ $i = 1 \dots 6$

$(p=2)$ f^* where $f^*(f) = 1$.

$C^0(K) = \langle v_1^*, \dots, v_5^* \rangle$, $C^1(K) = \langle e_1^*, \dots, e_6^* \rangle$, $C^2(K) = \langle f^* \rangle$.

What is $\delta_1 : C^0(K) \to C^1(K)$? Let's study $\delta_1(v_1^*) \in C^1(K)$.

$\delta_1(v_1^*)(e_1) = v_1^*(\partial e_1) = v_1^*(v_2 - v_1) = -1$
$\delta_1(v_1^*)(e_2) = v_1^*(\partial e_2) = v_1^*(v_3 - v_2) = 0$
$\delta_1(v_1^*)(e_3) = v_1^*(\partial e_3) = v_1^*(v_1 - v_3) = 1$
$\delta_1(v_1^*)(e_4) = v_1^*(\partial e_4) = v_1^*(v_5 - v_3) = 0$
$\delta_1^*(v_1^*)(e_5)$ and $\delta_1^*(v_1^*)(e_6) = 0$.

Thus, $\delta_1(v_1^*) = e_3^* - e_1^*$.

Check that $\delta_1(v_2^*) = e_1^* - e_2^*$ and $\delta_1(v_3^*) = e_2^* - e_3^* - e_4^* + e_6^*$.

So, $\delta_1(v_i^*)$ askes, which edges contain v_i ?

What is $\delta_2 : C^1(K) \to C^2(K)$?

$\delta_2(e_1^*) = \delta_2(e_2^*) = \delta_2(e_3^*) = 0$ since no face contains e_1, e_2 or e_3.
$\delta_2(e_4^*) = \delta_2(e_5^*) = \delta_2(e_6^*) = f^*$.

Let's check a couple of calculations.

$$\delta_2(e_1^*)(f) = e_1^*(\partial f) = e_1^*(e_4+e_5+e_6) = 0+0+0 = 0.$$
$$\delta_2(e_4^*)(f) = e_4^*(\partial f) = e_4^*(e_4+e_5+e_6) = 1+0+0 = 1.$$

Now for the cohomology groups.

$H^0(K) = \dfrac{\ker \delta_1}{0} = \ker \delta_1$. When is $\delta_1(\sum n_i v_i^*) = 0$?

$\delta_1(\sum n_i v_i^*) = \sum n_i \delta_1(v_i^*) =$

$n_1(e_3^* - e_1^*) + n_2(e_1^* - e_2^*) + n_3(e_2^* - e_3^* - e_4^* + e_5^*) + n_4(e_5^* - e_6^*) + n_5(e_4^* - e_5^*) =$

$(n_2-n_1)e_1^* + (n_3-n_2)e_2^* + (n_1-n_3)e_3^* + (n_5-n_3)e_4^* + (n_4-n_5)e_5^* + (n_3-n_4)e_6^* = 0$

$\Rightarrow \quad n_1 = n_2 = n_3 = n_5 = n_4.$

Thus, $\ker \delta_1 = \langle v_1^* + v_2^* + v_3^* + v_4^* + v_5^* \rangle \cong \mathbb{Z}$.

Hence $H^0(K) \cong \mathbb{Z}$.

$$H^1(K) = \frac{\ker \delta_2}{\operatorname{im} \delta_1}.$$

$$\operatorname{im}\left(\delta_1 : \underset{\mathbb{Z}^5}{\underset{\cong}{C^0(K)}} \to \underset{\mathbb{Z}^6}{\underset{\cong}{C^1(K)}}\right) = ?$$

cannot be onto!

$= \langle \delta_1(v_i^*) \rangle_{i=1\ldots5}$.

$$= \langle \underset{A}{e_3^* - e_2^*}, \underset{B}{e_1^* - e_2^*}, \underset{C}{e_2^* - e_3^* - e_4^* + e_6^*}, \underset{D}{e_4^* - e_5^*}, \underset{E}{e_5^* - e_6^*} \rangle$$

$C = -A - D - E$. So, we can eliminate it.

The others are L.I.

Thus $\operatorname{im} \delta_0 \cong \mathbb{Z}^4$.

$\operatorname{Ker} \delta_2 : C^1(K) \to C^2(K) = ?$

We know $e_1^*, e_2^*, e_3^* \to 0$ and $e_4^*, e_5^*, e_6^* \to f$.

Thus, $e_4^* - e_5^*, e_5^* - e_6^* \to 0$.

Thus,

$$\ker \delta_2 = \langle e_1^*, e_2^*, e_3^*, e_4^* - e_5^*, e_5^* - e_6^* \rangle \cong \mathbb{Z}^5.$$

$$H^1(K) = \frac{\langle e_1^*, e_2^*, e_3^*, e_4^* - e_5^*, e_5^* - e_6^* \rangle}{\langle e_3^* - e_2^*, e_1^* - e_2^*, e_4^* - e_5^*, e_5^* - e_6^* \rangle} \cong \frac{\langle e_1^*, e_2^*, e_3^* \rangle}{\langle e_3^* - e_1^*, e_1^* - e_2^* \rangle}$$

$$\cong \frac{\langle e_1^* - e_2^*, e_2^*, e_3^* - e_2^* \rangle}{\langle e_1^* - e_2^*, e_3^* - e_2^* \rangle} \cong \langle e_2^* \rangle \cong \mathbb{Z}.$$

$$H^2(K) = \frac{\ker \delta : C^2(K) \to 0}{\operatorname{im} \delta : C^1(K) \to C^2(K)} = \frac{C^2(K)}{C^2(K)} = 0.$$

$$\boxed{H^0(K) \cong \mathbb{Z},\ H^1(K) \cong \mathbb{Z},\ H^2(K) = 0.}$$