§47 Cohomology of CW complexes

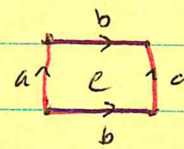
Thm 47.1 Let X be a CW complex. Let $\mathcal{D}(X)$ be it cellular chain complex. Then, $\forall p$,

$$H^p(\mathcal{D}(X)) \cong H^p(X).$$

Pf See textbook.

Ex Recall for T^2 we had

$$0 \to \underset{\langle e \rangle}{\overset{\shortparallel}{D_2(T^2)}} \xrightarrow{\partial_2} \underset{\langle a, b \rangle}{\overset{\shortparallel}{D_1(T^2)}} \xrightarrow{\partial_1} \underset{\langle v \rangle}{\overset{\shortparallel}{D_0(T^2)}} \to 0$$



$\operatorname{im}\partial_2 = 0$, $\ker\partial_2 = D_2$, $\operatorname{im}\partial_1 = 0$, $\ker\partial_1 = D_1$.

Thus, $H_2 = D_2/0 \cong \mathbb{Z}$, $H_1 = \frac{D_1}{0} \cong \mathbb{Z}^2$, $H_0 \cong \mathbb{Z}$.

Now consider the dual sequence

$$0 \leftarrow \underset{\mathbb{Z}}{\overset{\text{is}}{\operatorname{Hom}(D_2, \mathbb{Z})}} \xleftarrow{\delta_2} \underset{\mathbb{Z}^2}{\overset{\text{is}}{\operatorname{Hom}(D_1, \mathbb{Z})}} \xleftarrow{\delta_1} \underset{\mathbb{Z}}{\overset{\text{is}}{\operatorname{Hom}(D_0, \mathbb{Z})}} \leftarrow 0$$

For generator use

$\operatorname{Hom}(D_2, \mathbb{Z}) = \langle e^* \rangle$ where $e^*(e) = 1$.

$\operatorname{Hom}(D_1, \mathbb{Z}) = \langle a^*, b^* \rangle$ where $a^*(a) = 1$ $a^*(b) = 0$,

$\operatorname{Hom}(D_0, \mathbb{Z}) = \langle v^* \rangle$, $v^*(v) = 1$. $b^*(a) = 0$ $b^*(b) = 1$.

$(\delta_2 a^*)(e) = a^*(\partial e) = a^*(0) = 0$ Likewise $\delta_2 b^* = 0$

Thus,

$$H^2 = \frac{\operatorname{Hom}(D_2, \mathbb{Z})}{0} \cong \mathbb{Z}$$

$\ker \delta_2 = \mathrm{Hom}(D_1, \mathbb{Z})$.

$\delta_1(v^*)(a) = v^*(\partial a) = v^*(v - v) = 1 - 1 = 0.$

Likewise $\delta_1(v^*)(b) = 0.$

Thus, $\operatorname{im} \delta_1 = 0.$

$$H^1(T) = \frac{\ker \delta_2}{\operatorname{im} \delta_1} \cong \mathbb{Z}^2.$$

$$H^0(T) = \frac{\ker \delta_1}{0} = \frac{D_0}{0} \cong \mathbb{Z}.$$

Ex The Klein bottle K. Again $D_2 = \langle e \rangle$, $D_1 = \langle a, b \rangle$, $D_0 = \langle v \rangle$.
The dual seq is

$$0 \leftarrow \mathrm{Hom}(D_2, \mathbb{Z}) \xleftarrow{\delta_2} \mathrm{Hom}(D_1, \mathbb{Z}) \xleftarrow{\delta_1} \mathrm{Hom}(D_0, \mathbb{Z}) \leftarrow 0$$

$$H^2 = \frac{\mathrm{Hom}(D_2, \mathbb{Z})}{\operatorname{im} \delta_2}$$

$\delta_2(a^*)(e) = a^*(\partial e) = a^*(a + \cancel{b} + a + b) = 2 + 0 = 2$

$\delta_2(b^*)(e) = \qquad\qquad = 2.$

So, $\operatorname{im} \delta_2$ is $2 \times \mathrm{Hom}(D_2, \mathbb{Z}) = \langle 2e^* \rangle$.

Thus, $H^2(K) \cong \mathbb{Z}/2\mathbb{Z}$.

This is not the same as $H_2(K)$. Recall $H_2(K) = 0$.

$$H^1(K) = \frac{\ker \delta_2}{\operatorname{im} \delta_1}.$$

$\ker \delta_2 = \langle a^* - b^* \rangle \cong \mathbb{Z}.$

$\operatorname{im} \delta_1 = 0.$

Thus, $H^1(K) \cong \mathbb{Z}.$

This is not the same as $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$

You can check $H^0(K) \cong \mathbb{Z}.$

Ex Let $n > 0$. Then $H^i(S^n) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$

Pf See textbook, Cor. 47.2. As a CW complex S^n has two cells: a 0-cell and a n-cell. The entire bdy of the n-cell maps to the 0-cell.