§48

Cup Product (Singular)

Def Cup product. Let $\mathbb{Z}$ be the ring of integers (we could use any ring with a multiplicative unit). The cup product is a map

$$S^p(X) \times S^q(X) \xrightarrow{\cup} S^{p+q}(X)$$

defined as follows. Let $a^p \in S^p(X)$, $b^q \in S^q(X)$ then $a^p \cup b^q \in S^{p+q}(X)$ on a singular $p+q$-simplex T by

$$(a^p \cup b^q)(T) = \langle a^p \cup b^q, T \rangle =$$

$$\langle a^p, T \circ l(e_0, e_1, \ldots, e_p) \rangle \underset{\text{ring mult.}}{\cdot} \langle b^q, T \circ l(e_p, \ldots, e_{p+q}) \rangle.$$

Then this can be extended to all $p+q$-chains, $S_{p+q}(X)$.

We can restrict this to cocycles and if we can show cup products don't "see" coboundaries we induce a cup product on cohomology:

$$\cup : H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X).$$

If we let $H^*(X) = \oplus H^i(X)$ then $\cup$ determines a product on $H^*(X)$ and thus we have the cohomology ring of X.

Thm 48.1 On the cochain groups

(a). $\cup$ is bilinear

(b). $\cup$ is assoc.

(c) Let z^0 be the 0-cochain that is 1 on each sing. 0-simplex. Then z^0 acts as a unity element: $z^0 \cup a^p = a^p \cup z^0 = a^p$.

(d) $\delta(a^p \cup b^q) = (\delta a^p) \cup b^q + (-1)^p a^p \cup (\delta b^q)$.

We will prove (c) and illustraight (d).

Pf of (c)

$$\langle z^0 \cup a^p, T\rangle = \langle z^0, T \circ l(e_0)\rangle \cdot \langle a^p, T \circ l(e_0, \dots, e_p)\rangle$$

$$= 1 \cdot \langle a^p, T\rangle.$$

Thus $z^0 \cup a^p = a^p$. The proof that $a^p \cup z^0 = a^p$ is similar. $\square$

Ex for (d)

Let $p=3$, $q=2$. Let $T: \Delta_6 \to X$ be a sing. 6-simplex. Recall

$$\partial T = \sum_{i=0}^{6} (-1)^i T \circ l(e_0, \dots, \hat{e}_i, \dots, e_6).$$

We compute both sides of (d) and compare.

$$\langle \delta(a^3 \cup b^2), T \rangle = \langle a^3 \cup b^2, \partial T \rangle = \langle a^3 \cup b^2, \sum_{i=0}^{6} (-1)^i T \circ l(e_0, \dots, \hat{e}_i, \dots, e_6) \rangle$$

$$= \sum_{i=0}^{6} (-1)^i \langle a^3 \cup b^2, T \circ l(e_0, \dots, \hat{e}_i, \dots, e_6) \rangle$$

$$= \langle a^3, T \circ l(e_1, e_2, e_3, e_4) \rangle \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$

$$- \langle a^3, T \circ l(e_0, e_2, e_3, e_4) \rangle \cdot \langle b^2, T \circ l(e_4, e_5, e_6) \rangle$$

$$+ \langle a^3, T \circ l(e_0 e_1 e_3 e_4) \rangle \cdot \langle b^2, T \circ l(e_4, e_5, e_6) \rangle$$

$$- \langle a^3, T \circ l(e_0 e_1 e_2 e_4) \rangle \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$

$$+ \langle a^3, T \circ l(e_0 e_1 e_2 e_3) \rangle \cdot \langle b^2, T \circ l(e_3 e_5 e_6) \rangle$$

$$- \langle a^3, T \circ l(e_0, e_1 e_2 e_3) \rangle \cdot \langle b^2, T \circ l(e_3 e_4 e_6) \rangle$$

$$+ \langle a^3, T \circ l(e_0 e_1 e_2 e_3) \rangle \cdot \langle b^2, T \circ l(e_3 e_4 e_5) \rangle. \quad (0)$$

Now for the first term of the RHS. of (d).

$$\langle (\delta a^3) \cup b^2, T \rangle = \langle \delta a^3, T \circ l(e_0 e_1 e_2 e_3 e_4) \rangle \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$

$$= \langle a^3, \partial T \circ l(e_0 e_1 e_2 e_3 e_4) \rangle \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$

$$= \left(\sum_{i=0}^{4} (-1)^i \langle a^3, T \circ l(e_0, .. \hat{e}_i, .. e_4) \rangle \right) \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$

$$= \langle a^3, T \circ l(e_1 e_2 e_3 e_4) \rangle \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$

$$- \langle a^3, T \circ l(e_0 e_2 e_3 e_4) \rangle \cdot \langle \quad " \quad \rangle$$

$$+ \langle a^3, T \circ l(e_0 e_1 e_3 e_4) \rangle \cdot \langle \quad " \quad \rangle$$

$$- \langle a^3, T \circ l(e_0 e_1 e_2 e_4) \rangle \cdot \langle \quad " \quad \rangle$$

$$+ \langle a^3, T \circ l(e_0 e_1 e_2 e_3) \rangle \cdot \langle \quad " \quad \rangle. \qquad (*)$$

Now for the second term of the RHS of (d)

$$(-1)^3 \langle a^3 \cup (\delta b^2), T \rangle = - \langle a^3, T \circ l(e_0 e_1 e_2 e_3) \rangle \cdot \langle b^2, \partial T \circ l(e_3 e_4 e_5 e_6) \rangle$$

$$= - \langle a^3, T \circ l(e_0 e_1 e_2 e_3) \rangle \cdot \langle b^2, T \circ l(e_4 e_5 e_6) \rangle$$
$$+ \langle \quad " \quad \rangle \cdot \langle b^2, T \circ l(e_3 e_5 e_6) \rangle$$
$$- \langle \quad " \quad \rangle \cdot \langle b^2, T \circ l(e_3 e_4 e_6) \rangle$$
$$+ \langle \quad " \quad \rangle \cdot \langle b^2, T \circ l(e_3 e_4 e_5) \rangle. \qquad (\#)$$

Add (*) and (#). Two terms will cancel. Compare to (c). They are equal! $\square$

Thm 48.2 The cup product $\cup : S^p(X) \times S^q(X) \to S^{p+q}(X)$ induces a product on cohomology, also called the cup product,

$$\cup : H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X),$$

that is bilinear and assoc. The cohomology class $[z_0]$ acts as a unity element.

Pf All we have to do is show $\cup$ take pairs of cocycles to cocycles and that homologous cocycles go to homologous cocycles.

Suppose $z^p \in Z^p(X)$, $z^q \in Z^q(X)$. (Hence $[z^p] \in H^p(X)$ and $[z^q] \in H^q(X)$.) Claim: $z^p \cup z^q \in Z^{p+q}(X)$.

Pf

$$\begin{aligned}\delta(z^p \cup z^q) &= \delta z^p \cup z^q + (-1)^p z^p \cup \delta z^q \\ &= 0 \cup z^q \pm z^p \cup 0 = 0 \pm 0 = 0. \quad \square\end{aligned}$$

Suppose $z'^p \in [z^p]$, $z'^q \in [z^q]$. Then

$z'^p = z^p + \delta d^{p-1}$ and $z'^q = z^q + \delta f^{q-1}$ for some $d^{p-1} \in S^{p-1}(X)$, $f^{q-1} \in S^{q-1}(X)$.

We will show that $z'^p \cup z'^q \in [z^p \cup z^q]$.

$$z'^p \cup z'^q = z^p \cup z^q + \delta d^{p-1} \cup z^q + z^p \cup \delta f^{q-1} + \delta d^{p-1} \cup \delta f^{q-1}$$

$$= z^p \cup z^q + \delta\left(d^{p-1} \cup z^q + (-1)^p z^p \cup f^{q-1} + d^{p-1} \cup \delta f^{q-1}\right)$$

$$\in [z^p \cup z^q],$$

where we have used

$$\delta(d^{p-1} \cup z^q) = \delta d^{p-1} \cup z^q + \cancel{(-1)^{p-1} d^{p-1} \cup \overbrace{\delta z^q}^{=0}} = \delta d^{p-1} \cup z^q$$

$$\delta(z^p \cup f^{q-1}) = \cancel{\overbrace{\delta z^p}^{=0} \cup f^{q-1}} + (-1)^p (z^p \cup \delta f^{q-1}) = (-1)^p (z^p \cup \delta f^{q-1})$$

$$\delta(d^{p-1}, \delta f^{q-1}) = \delta d^{p-1} \cup \delta f^{q-1} + \cancel{(-1)^{p-1} (d^{p-1} \cup \overbrace{\delta\delta f^{q-1}}^{=0})} = \delta d^{p-1} \cup \delta f^{q-1}.$$

Finally, the role of z^0 is easy to check. $\square$

<u>Facts</u> ① $\cup$ is anti-commutative: $a^p \cup b^q = (-1)^{pq} b^q \cup a^p$.

② If $h: X \to Y$ is cont. then $h^*: H^*(X) \to H^*(Y)$ preserves the cup product: $h^*(a^p \cup b^q) = (h^* a^p) \cup (h^* b^q)$.

③ Cup products can be defined in relative cohomology as well.