§49 Cohomology Rings

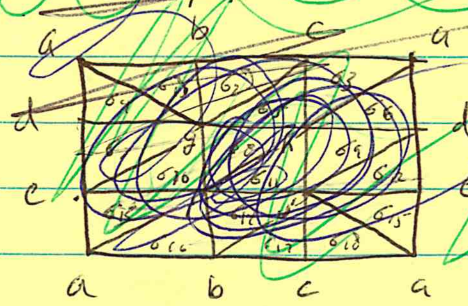
We work with simplicial homology. This will allow us to see the generators. Cup products can be defined for CW complexes but more algebraic machinery is involved. See the intro to Ch 3 of Hatcher's book.

To define the cup product for a simplicial complex K we have to first enumerate the vertices. Once this is done let

$$\langle a^p \cup b^q, [v_0, \dots, v_{p+q}] \rangle = \langle a^p, [v_0, \dots v_p] \rangle \cdot \langle b^q, [v_p, \dots, v_{p+q}] \rangle.$$

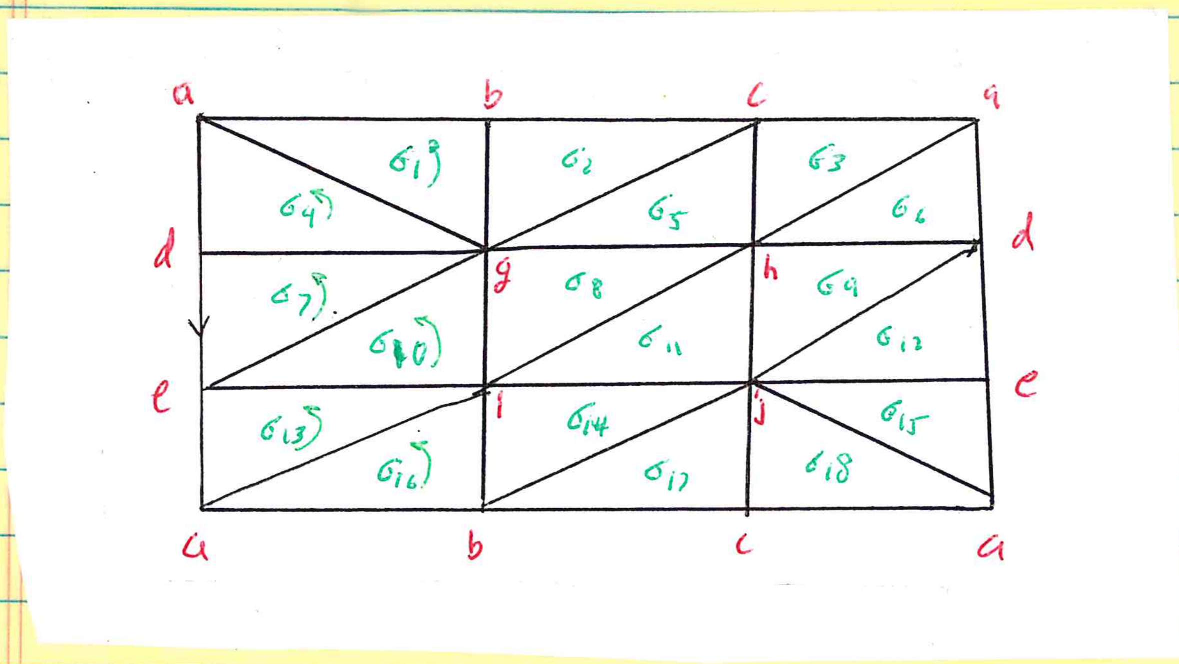
The theory goes through as before.

~~Ex Find the cohomology ring of T^2. Order the vertices alphabetically on the Δ complex below.~~



~~Orient all 2-simplicies counterclockwise.~~

Ex Find the cohomology ring of T^2. Order the vertices alphabetically on the complex below. Orient all 2-simplices counterclockwise.



Notice, as mention earlier, the cochain $[a,b]^*+[b,c]^*+[c,a]^*$ is <u>not</u> a cocycle: $\delta_1([a,b]^*+[b,c]^*+[c,a]^*) = \sigma_{16}^* - \sigma_1^* + \sigma_{17}^* - \sigma_2^* + \sigma_{18}^* - \sigma_5^* \neq 0$

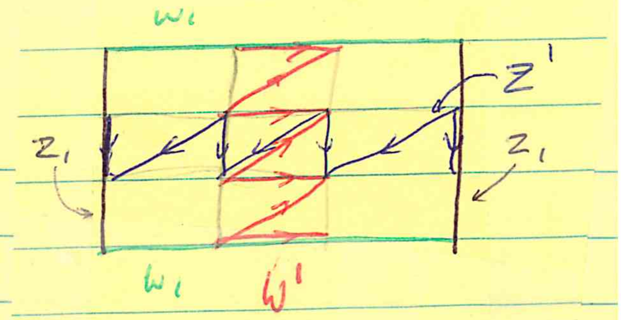
Let $z' = [d,e]^*+[d,j]^*+[h,j]^*+[h,i]^*+[g,i]^*+[g,e]^*$.

Then $\delta_1 z' = \sigma_7^* - \sigma_{12}^* + \sigma_{12}^* - \sigma_9^* + \sigma_9^* - \sigma_{11}^* + \sigma_{11}^* - \sigma_8^* + \sigma_8^* - \sigma_{10}^* + \sigma_{10}^* - \sigma_7^* = 0$

Let $w' = [b,c]^*+[g,c]^*+[g,h]^*+[i,h]^*+[i,j]^*+[b,j]^*$

Then $\delta w' = \cdots = 0$.

Why are these cocycles, generators of $H^1(T^2)$?
$[z'], [w']$



Recall $w_1 = [a,b] + [b,c] + [c,d]$
and $z_1 = [a,d] + [d,e] + [e,a]$
are generators of $H_1(T^2)$.

Compute:

$$\langle w^1, w_1 \rangle = 1 \qquad \langle w^1, z_1 \rangle = 0$$
$$\langle z^1, w_1 \rangle = 0 \qquad \langle z^1, z_1 \rangle = 1.$$

I'll do one of these:

$$\langle w^1, w_1 \rangle = \langle [b,c]^* + [g,c]^* + [g,h]^* + [i,h]^* + [i,j]^* + [b,j]^*, \; [a,b] + [b,c] + [c,d] \rangle$$

Now $[b,c]^*([b,c]) = 1$.
All the other pairs give zero. Thus, $\langle w^1, w_1 \rangle = 1$.

It follows that $[w^1]$ and $[z^1]$ are a basis for $H^1(T^2)$.
(See page 283-284.)

We know $H^2(T^2) \cong \mathbb{Z}$, but what can we use for a generator in the simplicial case? Notice $\delta_2(c) = 0$ for any 2-cochain c. Thus, any σ_i^* is a 2-cocycle and any two σ_i^*, σ_j^* are co-homologous. Thus, any $[\sigma_i^*]$ can be used as a generator of $H^2(T^2)$.
(This is dual to the case with H_0 where since $\partial : C_0 \to 0$ any vertex (for a path conn'd complex) can be taken as a generator for H_0.)

Let $\Lambda = [\sigma_1^*]$. Let $\alpha = [w^1]$ and $\beta = [z^1]$.

What about $H^0(T^2)$? Now, $\delta_0 : C^0 \to C^1$ and $\delta_0 v^* \neq 0$ for each $v^* \in \{a^*, b^*, \dots j^*\}$.
Let $z^0 = a^* + b^* + \cdots + j^*$, the sum of all covertices.
Then $\delta z^0 = 0$ since each edge appears twice with opposite sign. Then $[z^0]$ generates $H^0(T^2)$.

To understand the ring $H^*(T^2)$ we only need to compute the cup product of the generators. Since $[z^0]$ is the identity we know what it does. We need to fill out the table below.

$\cup$	α	β	Λ
α			
β			
Λ			

$\Lambda \cup A \in H^4(T^2) = 0$ so $\Lambda \cup \Lambda = 0$.

Likewise $\alpha \cup \Lambda$, $\Lambda \cup \alpha$, $\beta \cup \Lambda$, $\Lambda \cup \beta$ are zero.

What is $\alpha \cup \beta$? We find out by studying $w' \cup z'$ on each 2-simplex.

If no edge of $\partial \sigma_i$ is carried by $|w'|$ or $|z|$ then $\langle w' \cup z', \sigma_i \rangle = 0$. In fact an edge of $\partial\sigma_i$ must be carried by both to give a nonzero result. There are only two:

$$-\sigma_8 = [g\,h\,i] \text{ and } \sigma_{11} = [h\,i\,j]$$ (need to list them alphabetically)

Now,

$$\langle w' \cup z', [g\,h\,i] \rangle = \langle w', [g\,h] \rangle \cdot \langle z', [h\,i] \rangle = 1 \cdot 1 = 1.$$

$$\langle w' \cup z', [h\,i\,j] \rangle = \langle w', [h\,i] \rangle \cdot \langle z', [i\,j] \rangle = -1 \cdot 0 = 0.$$

Thus, $w' \cup z' = [g\,h\,i]^* = -\sigma_8^*$.

Thus, $\alpha \cup \beta = -\Lambda$.

Hence $\beta \cup \alpha = (-1)^{1\cdot 1} \alpha \cup \beta = \Lambda$. (Using anti-commutativity, see bottom page 289.)

Anti-commutativity also implies $\alpha \cup \alpha = -\alpha \cup \alpha$ and $\beta \cup \beta = -\beta \cup \beta$, Hence $\alpha \cup \alpha = \beta \cup \beta = 0$.

$\cup$	α	β	Λ
α	0	$-\Lambda$	0
β	Λ	0	0
Λ	0	0	0

$\square$

Ex Find Cohomology Ring of $T^2 \# T^2$.

The next page shows generators of $H^1(T^2 \# T^2)$. Let Y be the yellow one, R be the red one, B be the blue one, and G the green one. You can check these are 1-cocycles: each 2-simplex that appears in $\delta_1 Y, R, B$ or G, appears twice with opposite sign.

To see that the generate H^1 study how they act on generators for H_1. Let

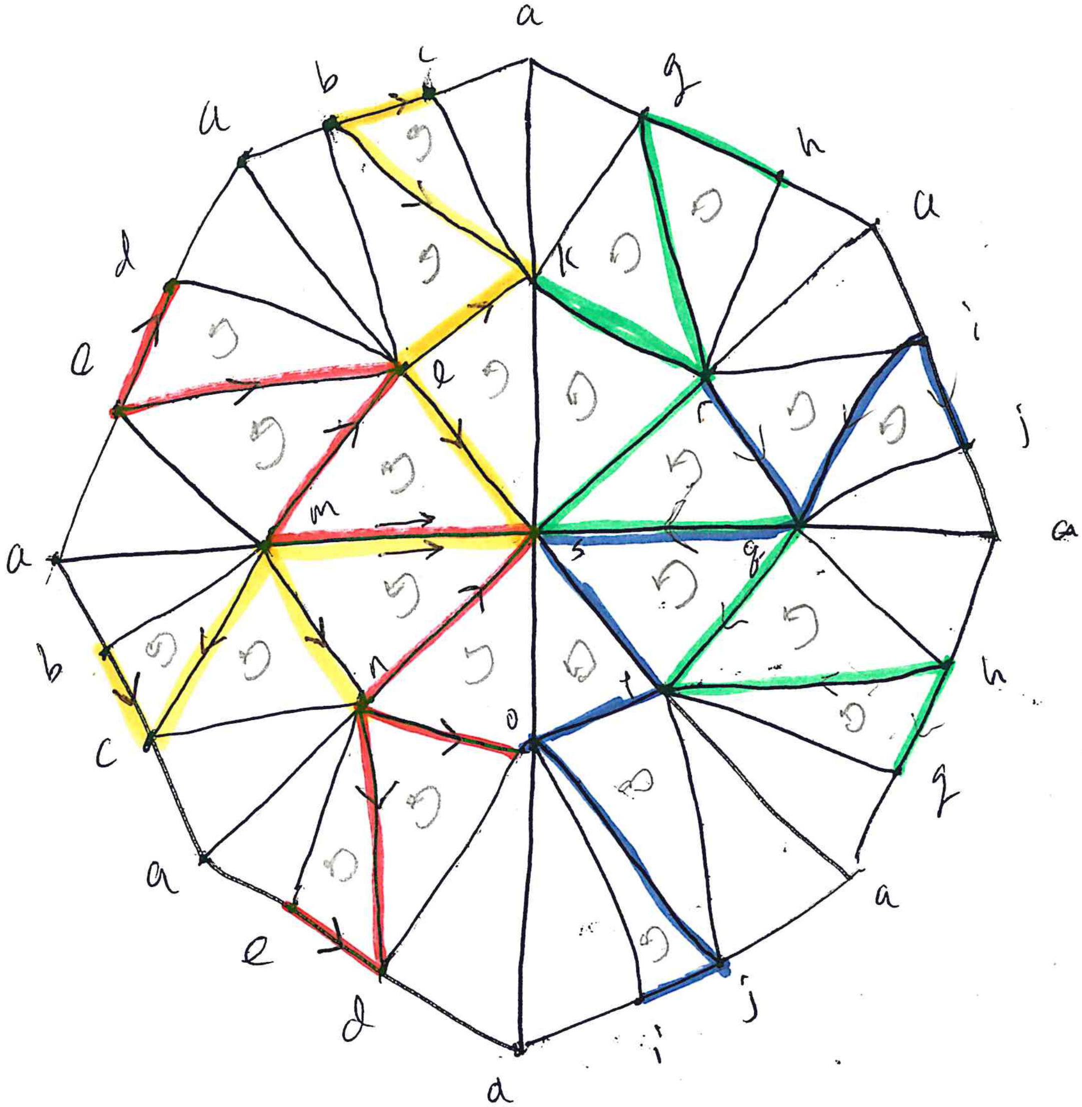
$$\alpha = [ab] + [bc] + [ca]$$
$$\beta = [ag] + [gh] + [ha]$$
$$\gamma = [ai] + [ij] + [ja]$$
$$\delta = [ae] + [ed] + [da]$$

Then we get

$\langle\ ,\ \rangle$	α	β	γ	δ
Y	1	0	0	0
B	0	1	0	0
G	0	0	1	0
R	0	0	0	1

Thus, $\{Y, B, G, R\}$ is a dual basis to $\{\alpha, \beta, \gamma, \delta.\}$

§48 3



As before the generator of H^0 is $[\Sigma \text{ all "vertices"}] = Z$.
Z is the identity element of the ring H^*.

For $H^2(T\#T)$ use $\Lambda =$ any 2-cosimplex oriented ccw.
Recall they are all cohomologous.

Since the cup product takes $H^1 \times H^2$, $H^2 \times H^1$ and $H^2 \times H^2$ to 0, the only products we care about are $H^1 \times H^1 \to H^2$.

Recall $H_2(T\#T)$ is generated by $[\Sigma \text{ all 2-simplices}]$.

We compute $Y \cup R$.

$$\langle Y \cup R, \sum_{all} \sigma_i \rangle = \langle Y \cup R, [lms] + [mns] \rangle$$

$$= \langle Y \cup R, [lms] \rangle + \langle Y \cup R, [mns] \rangle$$

$$= \langle Y, [lm] \rangle \cdot \langle R, [ms] \rangle + \langle Y, [mn] \rangle \cdot \langle R, [ns] \rangle$$

$$= 0 \cdot 1 + 1 \cdot 1 = 1$$

$[mns]$ is ccw so $Y \cup R = +\Lambda$.
Thus, $R \cup Y = -\Lambda$.

$$\langle G \cup B, \sum_{\text{all}} \sigma_i \rangle = \langle G \cup B, [qrs] + [pqs] \rangle$$

$$= \langle G \cup B, [qrs] \rangle + \langle G \cup B, [pqs] \rangle$$

$$= \langle G, [qr] \rangle \cdot \langle B, [rs] \rangle + \langle G, [pq] \rangle \cdot \langle B, [qs] \rangle$$

$$= \quad 0 \cdot 0 + (-1) \cdot 1 = -1.$$

Thus, $G \cup B = -\Lambda$ and $B \cup G = \Lambda$.

It is easy to see $Y \cup B = Y \cup G = R \cup B = R \cup G = 0$ and we know $G \cup G = Y \cup Y = B \cup B = R \cup R = 0$.

Thus,

$\cup$	Y	R	B	G
Y	0	Λ	0	0
R	$-\Lambda$	0	0	0
B	0	0	0	Λ
G	0	0	$-\Lambda$	0

gives $H^*(T^2 \# T^2)$.