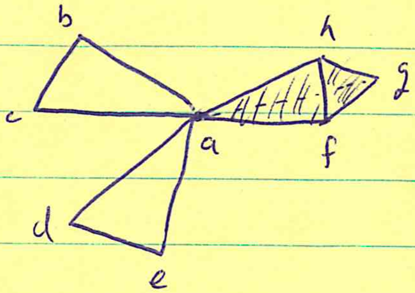
Example 4 Consider the simplicial complex below.



It is homeomorphic to $S^1 \vee S^1 \vee S^2$. It homology and cohomology groups are...

$$H_0 \cong \mathbb{Z}, \; H_1 \cong \mathbb{Z}^2, \; H_2 \cong \mathbb{Z}$$

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The homology groups can be computed quickly uses the Mayer-Vietoris sequence (§25). The cohomology groups can be found the same way (there is a M-V seq in cohomology although we haven't covered this), or by showing they must be isomorphic to the homology groups using obvious CW complex. This is how the textbook does it.

Hence, either cohomology or homology distinguishes $S^1 \vee S^1 \vee S^2$ from $T^2 = S^1 \times S^1$. But the cohomology ring does.

We claim the cohomology ring of $S^1 \vee S^1 \vee S^2$ is trivial.

Let $w_1 = [a\,b] + [b,c] + [c,a]$

and $z_1 = [a\,d] + [d,e] + [e,a]$.

Those generate H_1.

Let $w' = [b,c]^*$ and $z' = [d,e]^*$. Then

$$\langle w', w_1 \rangle = 1 \qquad \langle w', z_1 \rangle = 0$$
$$\langle z', w_1 \rangle = 0 \qquad \langle z', z_1 \rangle = 1.$$

They are cocycles: $\delta_1 w' = \delta_1 z' = 0$. (There are no 2-simplicies that have $[b,c]$ or $[d,e]$ as edges.)

But $w' \cup z'$, is trivial since no 2-simplex has an edge carried by them. By anti-commutativity $z' \cup w'$, $w' \cup w'$, $z' \cup z'$ are zero also.

Since the cohomology ring of T^2 is not trivial, we have distinguished it from $S^1 \vee S^1 \vee S^2$.

<u>Note</u> $\pi_1(S^1 \vee S^1 \vee S^2)$ = a free group on two generators while $\pi_1(T^2)$ is abelian. It can also be shown that

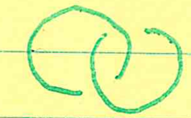
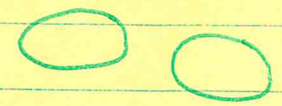
$$\pi_2(S^1 \cup S^1 \cup S^2) \cong \mathbb{Z}$$

while

$$\pi_2(T^2) = 0.$$

$\blacksquare$

Exercise 8 Let A be the union of two once-linked circles in S^3 as shown and let B be the union of two unlinked circles in S^3.

A  B 

Show that while the homology and cohomology groups of S^3-A and S^3-B are isomorphic, the cohomology rings are not.

Solution I will show "in class" (in a separate video) that there is a deformation retraction from S^3-A to a torus while the is a deformation retraction from S^3-B to a space homotopic to $S^1 \vee S^1 \vee S^2$. Following Example 4 this solves the problem.

Note S^3-A and S^3-B can also be distinguished by π_2.

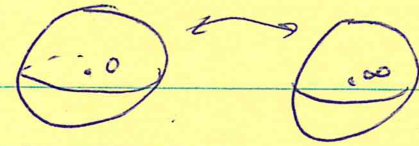
$$\pi_2(S^3-A) = 0 \quad , \quad \pi_2(S^3-B) \cong \mathbb{Z}.$$

They also have different fundamental groups:

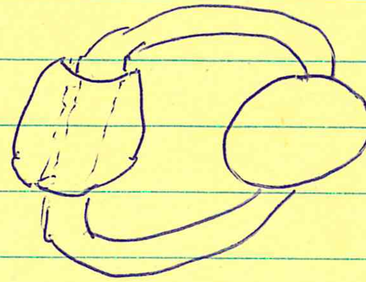
$$\pi_1(S^3-A) \cong \mathbb{Z}^2 \qquad \pi_1(S^3-B) = \text{free group on two symbols.}$$

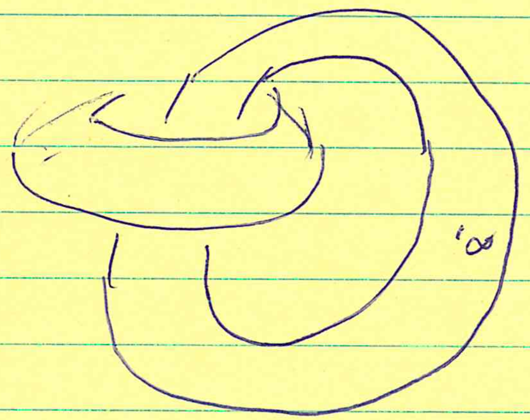
$S^3 - A$ has a def. rect. to T^2.

$S^3 = B^3 \cup_\partial B^3$

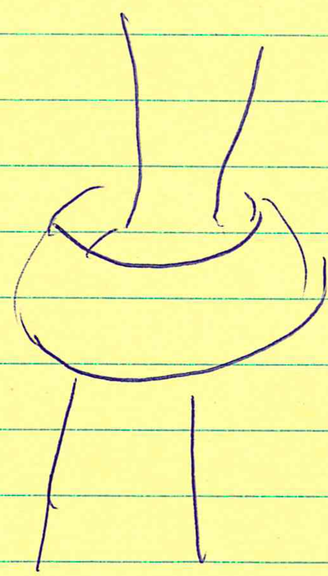


$S^3 = (S^1 \times D^2) \cup_\partial (S^1 \times D^2)$





There is a def. rect. from $(S^1 \times D^2) - (S^1 - 0)$ to $S^1 \times S^1$, since $\exists$ a def. rect. from $D^2 - 0$ to S^1.

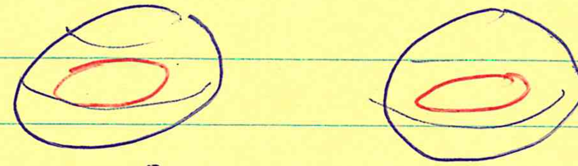


Do this on each ~~to~~ solid torus − core and we get a def. rect. from $S^3 - A$ to an embedded $S^1 \times S^1$.

Hence the cohom. ring of $S^3 - A$ is ~~the same~~ iso. to the cohom. ring of T^2.

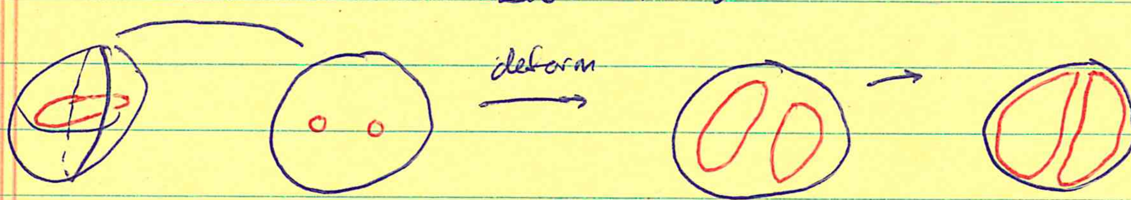
$S^3 - B$ has a def. retraction to a space homotopic to $S^1 \vee S^1 \vee S^2$.

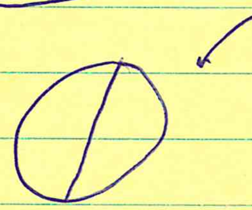
Let $S^3 = B^3 \cup_\partial B^3$ with ~~one~~ circle of B in each B^3



B^3 B^3

Deform each circle into a thin solid torus staying inside the B^3's. Look at a slice:

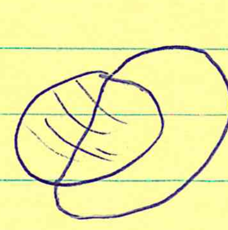


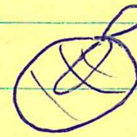


Do this to each slice to get

$S^2 \cup$ a axis.

We have two of these. Gluing the S^2's back together gives:



Then homotope this to 

which is homeo. to  $= S^1 \vee S^1 \vee S^2$.

Thus, the cohomology ring of $S^3 - B$ is trivial.