§51 Homology with Arbitrary Coefficients

Let $\mathcal{C} = \{C_p, \partial\}$ be a chain complex. Let G be an abelian group. We define a new chain complex by

$$\mathcal{C} \otimes G = \{C_p \otimes G, \partial \otimes id\}.$$

Since $\partial \otimes id \circ \partial \otimes id = \partial \circ \partial \otimes id = 0 \otimes id = 0$, $\mathcal{C} \otimes G$ is indeed a chain complex. Its p^{th} homology group is denoted

$$H_p(\mathcal{C}; G)$$

and is called the p^{th} homology group of $\mathcal{C}$ with coefficients in G. (We will see the connection with the definition in §10 later.) Notice that $H_p(\mathcal{C}; \mathbb{Z}) = H_p(\mathcal{C})$.

Reduced homology groups with coeff's in G are also defined. See textbook. We again have $H_0(\mathcal{C}; G) \cong \tilde{H}_0(\mathcal{C}; G) \oplus G$ and $H_p(\mathcal{C}; G) = \tilde{H}_p(\mathcal{C}; G)$, $p \neq 0$.

Now we look at chain maps and just list some facts.

Let $\phi: \mathcal{C} \to \mathcal{D}$ be a chain map.
Then $\phi \otimes id : \mathcal{C} \otimes G \to \mathcal{D} \otimes G$ is also a chain map.

The induced map on homology is denoted ϕ_* instead of $(\phi \otimes id)_*$.

If $\phi, \psi : \mathcal{C} \to \mathcal{D}$ are chain maps and D is a chain homotopy between them, then $D \otimes id$ is a chain homotopy between $\phi \otimes id$ and $\psi \otimes id$. Thus if ϕ and ψ are chain homotopic, then ϕ_* and ψ_* are equal ~~for~~ for all choices of coeff's.

Let $0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to 0$ be a short exact seq of chain complexes that splits in each dim.
Then

$$0 \to C_p \otimes G \to D_p \otimes G \to E_p \otimes G \to 0$$

is exact. (Thm 50.4)
We can apply the zig-zag lemma to get

$$\cdots \to H_p(\mathcal{C}; G) \to H_p(\mathcal{D}; G) \to H_p(\mathcal{E}; G) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}; G) \to \cdots$$

exact.

Now we take a quick look at the three homology theories we have: simplicial, singular, and cellular (CW).

Simplicial Let (K, K_0) be a simplicial pair. We have the chain complex $\mathcal{C}(K, K_0)$ and we define

$$H_p(K, K_0; G) = H_p(\mathcal{C}(K, K_0); G)$$

and similarly for reduced homology.

Since $C_p(K, K_0)$ is free abelian, $C_p(K, K_0) \otimes G$ is a direct sum of copies of G one for each p-simplex of K not in K_0. Orient the p-simplices and $C_p(K, K_0) \otimes G$ is generated by $\sigma_\alpha \otimes g_\alpha$, in fact every member of $C_p(K, K_0) \otimes G$ has a unique expression

$$\sum_{\text{finite}} \sigma_i \otimes g_i$$

The define in §10 has $c_p \in C_p(K, K_0; G)$ ~~generated by~~ represented by $\sum g_i \sigma_i$. The boundary maps do the same things:

$$\partial\left(\sum \sigma_i \otimes g_i\right) = \sum \partial\sigma_i \otimes g$$

$$\partial\left(\sum g_i \sigma_i\right) = \sum g\, \partial\sigma_i$$

So, the two definitions of ~~homo~~ simplicial homology group over G are isomorphic.

One can verify that $H_*(K, K_0; G)$ satisfies all the properties of a homology theory (the axioms).

<u>Singular</u> Very similar but the excision axiom is harder to check. See textbook.

<u>Cellular</u> Ditto.