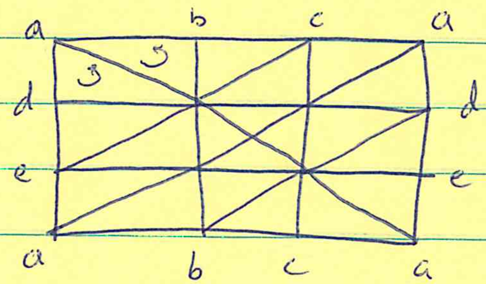
Section 6 Homology Groups of Surfaces.

Thm 6.2 Homology groups of a torus. Let T be the simplicial complex below



Orient all the 2-simplicies counterclockwise. Orient the edges arbitrarily. Then

$$H_0(T) \cong \mathbb{Z} \quad H_1(T) \cong \mathbb{Z}^2 \quad H_2(T) \cong \mathbb{Z}.$$

Pf $H_0(T) \cong \mathbb{Z}$ because T is connected.

We let

$$\gamma = \sum_{\text{all 2-simplicies}} \sigma_i$$

$$w = [a,b] + [b,c] + [c,a]$$

$$z = [a,d] + [d,e] + [e,a].$$

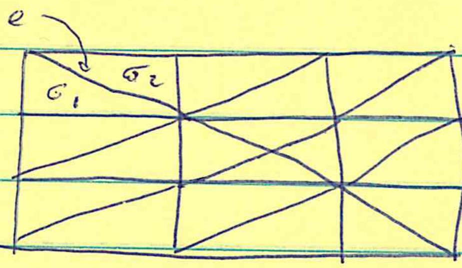
Let A be the subcomplex with $|A| = |z| + |w|$.
We will establish the following facts.

1. Every 1-cycle of T is homologous to a 1-cycle carried by A.
3. Every 1-cycle of A is of the form $nw + mz$.
2. If d is a 2-chain s.t. ∂d is carried by A then $d = n\gamma$.
4. $\partial\gamma = 0$.

(4) $\partial \gamma = 0$ is clear since every edge cancels out.

(2) Let $d = \sum_{\text{all } \sigma_i} n_i \sigma_i$ be a 2-chain s.t. ∂d is carried by A. We need to show all the n_i's are equal.

Start with σ_1 and σ_2 shown below. They have an edge e in common that is not in A. It follows that $n_1 = n_2$.

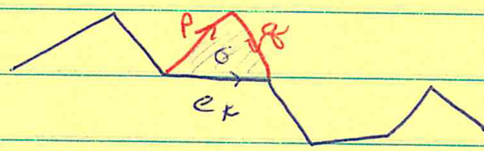


Continue this until all the n_i are equal. Thus

$$d = \sum n_i \sigma_i = n_1 \sum \sigma_i = n_1 \gamma.$$

By (4) we now know that $\partial d = 0$.

(1) Let $c = \sum_{i=1}^{m} n_i e_i$ be any 1-chain. Suppose e_k is an edge of a 2-simplex σ, whose other two edges are p and q as shown:

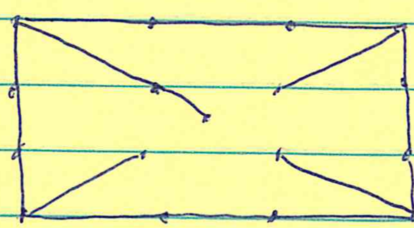


Then $c \sim c' = \sum_{\substack{i=1 \\ i \neq k}}^{m} n_i e_i + n_k p + n_k q$,

since $c - c' = n_k(e_i - p - q) = n_k \partial \sigma$.

Note: Edge p and q may appear elsewhere in c.

This shows (or can be used to show) any 1-chain in T is homologous to a 1-chain carried by the ~~sub~~ sub complex below.



Now assume c is a 1-cycle carried by this subcomplex. Since $\partial c = 0$, c cannot have a vertex that meets only one edge. Hence c is carried by the subcomplex A.

3 Let c be a 1-cycle in A,

$$c = n_1[a,b] + n_2[b,c] + n_3[c,a] + n_4[a,d] + n_5[d,e] + n_6[e,a].$$

$$\partial c = (-n_1 + n_3 - n_4 + n_6)a + (n_1 - n_2)b + (n_2 - n_3)c$$

$$+ (n_4 - n_5)d + (n_5 - n_6)e = 0.$$

Thus, $n_1 = n_2 = n_3$ and $n_4 = n_5 = n_6$.
Let $n = n_1$, $m = n_4$. Then $c = nw + mz$.

This means $H_1(T)$ is generated by w and z (technically their cosets), but will need to prove there are no relations.

Suppose $nw + mz = \partial(\overset{d}{\text{2-chain}})$. By 2 $d = k\gamma$ and by 4 $\partial d = 0$. Thus $nw + mz = 0$. Hence $n = m = 0$.
Thus,

$$H_1(T) = \langle w, z \rangle \cong \mathbb{Z}^2.$$

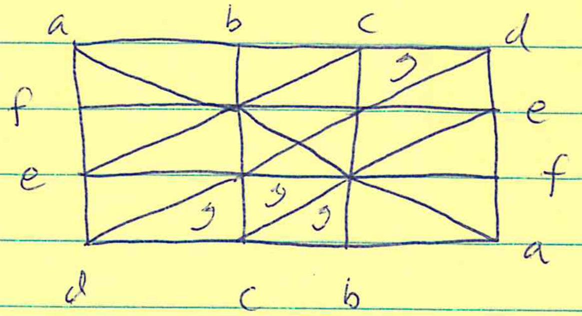
$H_2(T)$ is easy.

Let d be a 2-cycle. Then $\partial d = 0$, which is carried by A, so $d = n\gamma$. Thus $Z_2(T) = \langle \gamma \rangle$. Since $B_2(T) = 0$ we have

$$H_2(T) = \langle \gamma \rangle \cong \mathbb{Z}.$$

Thm 6.4 Homology group of a projective plane.

Let P be the complex below.



Orient all the 2-simplices counterclockwise
Orient the edges arbitrarily. Then

$$H_0(P) \cong \mathbb{Z} \quad H_1(P) \cong \mathbb{Z}/2\mathbb{Z} \quad H_2(P) = 0.$$

Pf Let $\gamma = \sum \sigma_i$

Let $z = [a,b] + [b,c] + [c,d] + [d,e] + [e,f] + [f,a]$.

Let A be the subcomplex s.t. $|A| = |z|$.

(1) Every 1-cycle of P is homologous to a 1-cycle carried by A. The proof is ~~the~~ the same as in Thm 6.2.

(2) If d is a 2-chain with ∂d carried by A, then $d = n\gamma$. The proof is the same as in Thm 6.2

(3) Every 1-cycle carried by A is a multiple of z.

Pf: Let $c = n_1[a,b] + n_2[b,c] + n_3[c,d] + n_4[d,e] + n_5[e,f] + n_6[f,a]$

Then $\partial c = (n_6 - n_1)a + (n_1 - n_2)b + (n_2 - n_3)c + (n_3 - n_4)d$
$+ (n_4 - n_5)e + (n_5 - n_6)f.$

Thus, $\partial c = 0$ iff $n_1 = n_2 = n_3 = n_4 = n_5 = n_6$.

(4) $\partial \gamma = -2z$.

pf: Just do the calculation.

(1) + (3) $\Rightarrow$ Every 1-cycle of P is homolgous to a 1-cycle of the form nz. Thus, $Z_1(P) = \langle z \rangle$.

But $2z = \partial(-\gamma)$ and thus bounds, i.e., $2z \in B_1(P)$. Thus, $H_1(P) \cong \mathbb{Z}/2\mathbb{Z}$ or is trivial. Could z bound?

Suppose $z = \partial d$ for some 2-chain. Then $d = n\gamma$. Then $z = -2nz$. (= not $\sim$). This is impossible. Thus,

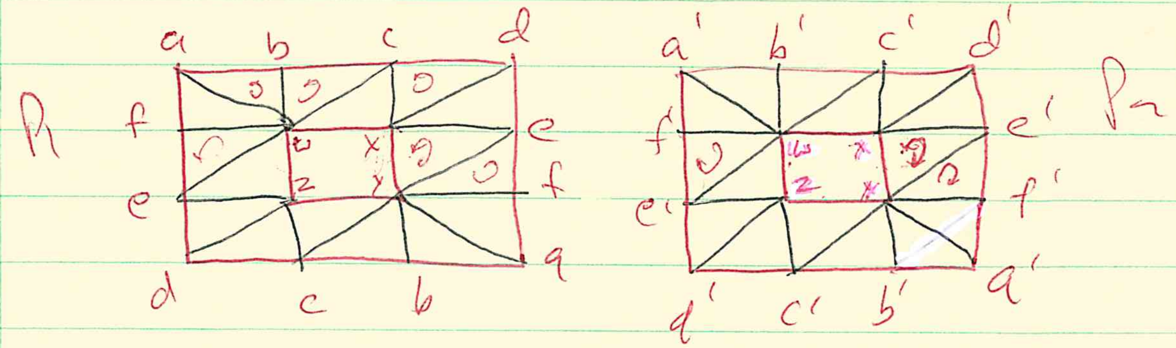
$$H_1(P) = \frac{\langle z \rangle}{\langle 2z \rangle} \cong \mathbb{Z}/2\mathbb{Z}.$$

Now for $H_2(P)$. Claim $Z_2(P) = 0$. Let d be a 2-cycle with $\partial d = 0$. Then $d = n\gamma \Rightarrow \partial d = -2nz$. But $-2nz \neq 0$ unless $n = 0$. Thus, $d = 0$ is the only 2-cycle. Hence,

$$H_2(P) = 0.$$

Since P is connected, $H_0(P) \cong \mathbb{Z}$. $\square$

Thm 6.5 Let $P \# P$ be the connected sum of two projective planes.



Then: $H_1(P\#P) = \mathbb{Z} \oplus \mathbb{Z}_2$ $H_2(P\#P) = 0$

Proof Let $\gamma = \sum_{\text{all}} \sigma$. Orient all σ counter-clockwise in P_1, σ clockwise in P_2.

Let $w = [ab] + [be] + \cdots + [fa]$

Let $z = [a'b'] + \cdots + [f'a']$

Let A be the subcomplex with $|A| = |w| \cup |z|$.

① & ② still hold.

③ Every 1-cycle carried by A is of the form $mw + nz$. (same proof)

④ $\partial\gamma = -2w + 2z$. (watch gluing!)

Claim $Z_2 = 0$

H_2: Let $\partial d = 0$. $d = n\gamma$ $\partial d = n(-2w + 2z) \neq 0$,
2-chain unless $n = 0$.

Thus any 2-cycle is trivial. $Z_2 = 0 \Rightarrow H_2 = 0$.

H_1: Every 1-cycle is homologous to $mw + nz$
Are there any other relations?
Yes $2z - 2w = \partial\gamma \Rightarrow 2z - 2w \sim 0$.
Are others?

$$nw + nz \sim 0 \Rightarrow mw + nz = \partial d = p\partial\gamma$$

$$= p(2z - 2w) \Rightarrow -m = n = 2p,$$

So, nothing new.

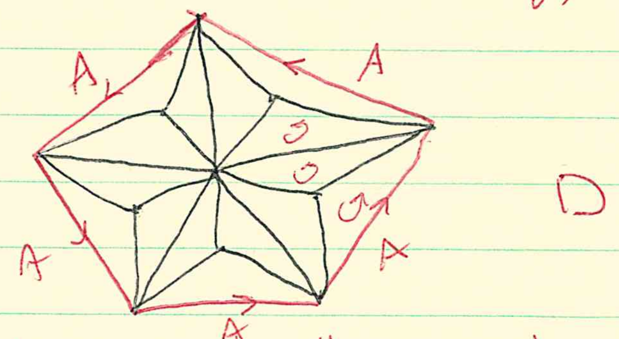
$$H_1 \cong \langle w, z \mid 2z - 2w = 0 \rangle \cong \frac{\mathbb{Z}^2}{\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \mathbb{Z}^2}$$

$$\cong \langle z - w, z \mid 2(z - w) = 0 \rangle$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \mathbb{Z} \oplus \mathbb{Z}_2.$$

Exercise 6 (a) Compute the homology groups of:



Let $\gamma = \sum \sigma$. Orient all σ counter-clockwise. Orient all A as shown. Orient other edge randomly. The following are facts:

① Every 1-cycle of D is homologous to a 1-cycle of A. : same proof

② if ∂d is carried by A $d = n\gamma$: same proof

③ Every 1-cycle of A is a of A : obvious

④ $\partial\gamma = 5A$: easy

H_1: ①+③ $\Rightarrow$ every 1-cycle of D is homologous to some nA. But by ④ $5A \sim 0$. Are there any other relations? Suppose $\partial d = mA$. (ie $mA \sim 0$) Then

$d = n\gamma \Rightarrow \partial d = n\partial\gamma = n5A \Rightarrow mA = 5nA$

So m must be a multiple of 5. No other relations. Thus $H_1 = \mathbb{Z}_5$.

H_2: $B_2 = 0$. Claim $Z_2 = 0$. Pf: $\partial d = 0 \Rightarrow d = n\gamma$

$\Rightarrow \partial d = n\partial\gamma = n5A = 0$ iff $n = 0$. So $Z_2 = 0$.

Thus $H_2 = 0$.

Thm 6.3 shows that for a simplicial complex for the Klein bottle K we get $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $H_2(K) = 0$, just like for $P^2 \# P^2$ (Thm 6.5).

As you may know K and $P^2 \# P^2$ are homeomorphic. This is covered in Munkres' Topology, Chapter 12. It is an exercise on page 454. It is part of the general classication of surfaces.

Thm Every compact, connected 2-manifold without boundary is homeomorphic to one and only one of the surfaces listed below.

orientable: S^2, T^2, $T^2 \# T^2$, $T^2 \# T^2 \# T^2$, ...
non orientable: P^2, $P^2 \# P^2$, $P^2 \# P^2 \# P^2$, ...

Their homology groups are in Exercise 4, Section 6 of Munkres' ~~Intro to~~ Elements of Alg. Top.

$$H_1(S^2) = 0 \qquad H_2(S^2) \cong \mathbb{Z}$$
$$H_1(T_1^2 \# \cdots \# T_n^2) \cong \mathbb{Z}^{2n} \qquad H_2(T_1^2 \# \cdots \# T_n^2) \cong \mathbb{Z}$$
$$H_1(P_1^2 \# \cdots \# P_k^2) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z} \qquad H_2(P_1^2 \# \cdots \# P_k^2) = 0.$$

Fact $T_1^2 \# \cdots \# T_n^2 \# P^2 \approx P_1^2 \# \cdots \# P_{2n+1}^2$.

Note: The dunce caps are not manifolds.