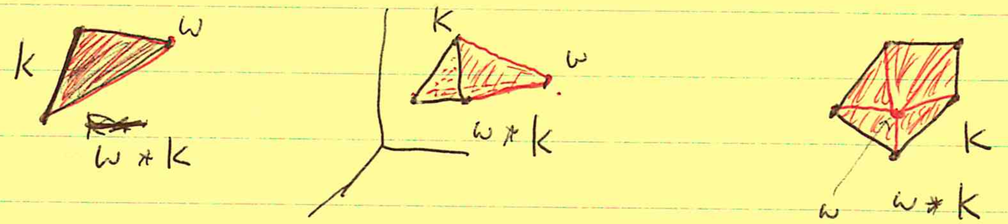
Section 8 Cones

Def Let K be an n-dim. complex in $\mathbb{R}^m$, $m > n$. Let $w \in \mathbb{R}^m$ be s.t. every ray from w meets $|K|$ in at most one point. Let

$$w * K = \{[w, a_0, \dots, a_p] \mid [a_0, \dots, a_p] \in K\} \cup \{K\} \cup \{w\}.$$

Then $w * K$ is an $n+1$-dim. complex called the cone with base K and vertex w.



Notation For $\sigma = [a_0, \dots, a_p]$, let $[w, \sigma] = [w, a_0, \dots, a_p]$. If $c = \sum n_i \sigma_i$ is a p-chain, let $[w, c] = \sum n_i [w, \sigma_i]$. This bracket opperation,

$$[w, -] : C_p(K) \to C_p(w * K)$$

is a group homomorphism.

Lemma ① $$\partial [w, \sigma] = \begin{cases} \sigma - w & \text{if } \dim \sigma = 0 \\ \sigma - [w, \partial \sigma] & \text{if } \dim \sigma > 0. \end{cases}$$

② Let $c_0 \in C_0(K)$ and $c_p \in C_p(K)$. Then

$$\partial [w, c_0] = c_0 - \varepsilon(c_0) w \quad \left(\varepsilon\left(\sum n_i v_i\right) = \sum n_i\right)$$

$$\partial [w, c_p] = c_p - [w, \partial c_p], \quad p > 0.$$

Pf If $\dim \sigma = 0$, then σ is a vertex, so $[w, \sigma]$ is an edge. Hence $\partial [w, \sigma] = \sigma - w$.

Let $c_0 = \sum n_i v_i$. Then $\partial [w, c_0] = \partial \sum n_i [w, v_i]$

$= \sum n_i \partial [w, v_i] = \sum n_i (v_i - w) = \sum n_i v_i - \sum n_i w$

$= c_0 - \varepsilon(c_0) w$.

Let $\sigma = [a_0, .., a_p]$, $p > 0$. Then $\partial [w, \sigma] = \partial [w, a_0, \dots a_p]$

$= [a_0, \dots, a_p] + \sum_{i=0}^{p} (-1)^{i+1} [w, a_0, \dots, \hat{a}_i, \dots a_p]$

$= \sigma - [w, \sum_{i=0}^{p} (-1)^i [a_0 \dots \hat{a}_i \dots a_p]] = \sigma - [w, \partial \sigma]$.

Let $c_p = \sum n_i \sigma_i \in C_p(K)$, $p > 0$. Then

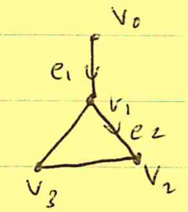
$\partial [w, c_p] = \partial \left(\sum n_i [w, \sigma_i] \right) = \sum n_i \partial [w, \sigma_i]$

$= \sum n_i (\sigma_i - [w, \partial \sigma_i]) = \sum n_i \sigma_i - \sum n_i [w, \partial \sigma_i]$

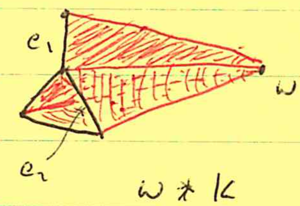
$= c_p - [w, \sum n_i \partial \sigma_i] = c_p - [w, \partial \sum n_i \sigma_i]$

$= c_p - [w, \partial c_p]$.

Ex



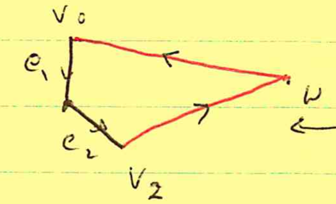
K



$w * K$

Let $c_1 = e_1 + e_2$.
$\partial c_1 = v_2 - v_0$

$\partial [w, c_1] = c_1 - [w, \partial e_1] = c_1 - [w\, v_2] + [w, v_0]$
$= e_1 + e_2 + [v_2, w] + [w, v_0]$



Thm 8.2 $\quad \tilde{H}_p(w * K) = 0, \ \forall p.$ $\quad$ [We say $w * K$ is acyclic when this happens.]

Pf $\quad$ The proof is just a calculation using the lemma. It is not deep, but is good practice.

$\tilde{H}_0(w * K) = 0$ since $|w * K|$ is connected even if $|K|$ is not.

Let $p > 0$. We claim $Z_p(w * K) = 0$. Let $z_p \in Z_p(w * K)$. We will find a $p+1$-chain in $w * K$ with boundary z_p.

If z_p is carried by K then $\partial [w, z_p] = z_p - [w, \partial z_p]$ $= z_p - [w, 0] = z_p$ and we are done. If not, let c_p be the part of z_p carried by K. We claim, $z_p = \partial [w, c_p]$. Before proving this observe that in the example above

$$e_1 + e_2 + [v_2, w] + [w, v_0] = \partial [w, e_1 + e_2].$$

For each simplex σ of $z_p - c_p$, $\exists$ a $p-1$-simplex of K, σ', with $\sigma = [w, \sigma']$ by definition. Thus,

$\exists\ d_{p-1} \in C_{p-1}(K)$ s.t. $z_p - c_p = [w, d_{p-1}]$.

Now $z_p - \partial[w, c_p] = c_p + [w, d_{p-1}] - (c_p - [w, \partial c_p])$

$= [w, d_{p-1}] + [w, \partial c_p] = [w, d_{p-1} + \partial c_p]$.

Let $e_{p-1} = d_{p-1} + \partial c_p$. Thus $z_p - \partial[w, c_p] = [w, e_{p-1}]$. $(\#)$

Next $\partial[w, e_{p-1}] = \partial z_p - \partial\partial[w, c_p] = 0 - 0 = 0$.

By the lemma,

$$\left.\begin{array}{l} \text{if } p=1, \quad e_{p-1} - \varepsilon(e_{p-1})w = 0, \text{ and} \\ \text{if } p>1, \quad e_{p-1} - [w, \partial e_{p-1}] = 0. \end{array}\right\} (\star)$$

But e_{p-1} is carried by K and w and no part of $[w, \partial e_{p-1}]$ is carried by K. Thus, the terms in $(\star)$ are independent. Thus, $e_{p-1} = 0$.

But $(\#) \Rightarrow z_p - \partial[w, c_p] = [w, 0] = 0 \Rightarrow z_p = \partial[w, c_p]$.

Thus, $z_p \in B_p$ and so $H_p = Z_p / B_p = 0$, $p > 1$.

$\boxed{\star}$

Thm 8.3 Let σ be an n-simplex.

(a) Let K_σ be the complex ~~consisting~~ consisting of σ and all faces of σ. Then K_σ is acyclic.

(b) Assume $n \geq 1$. Let Σ_σ be the complex of all faces of σ. Then

$$\tilde{H}_p(\Sigma_\sigma) \cong \begin{cases} \mathbb{Z} & p = n-1 \\ 0 & \text{otherwise} \end{cases}$$

In the $p=n-1$ case, $\tilde{H}_p(\Sigma_\sigma)$ is generated by $\partial\sigma$. (Assume σ has been given an orientation.)

pf (a) If $n=0$, $|K_\sigma|$ is a point and so $\tilde{H}_0(K_\sigma)=0$. For $n \geq 1$, K_σ is a cone using any vertex for w and the opposite face for the base. We know cones are acyclic.

(b) If $n=1$, then K_σ is an edge with its end points. Then $|\Sigma_\sigma|$ is two points so $\tilde{H}_0(\Sigma_\sigma) \cong \mathbb{Z}$.

Suppose $n>1$. We study the chain groups for K_σ and Σ_σ. They are the same except for the top dimension, n.

$$\begin{tikzcd}[column sep=small]
\langle\sigma\rangle \arrow[d, phantom, "\shortparallel"] & & & & & & & & \\
C_n(K_\sigma) \arrow[r, "\partial_n"] & C_{n-1}(K_\sigma) \arrow[r, "\partial_{n-1}"] \arrow[d, phantom, "\shortparallel"] & C_{n-2}(K_\sigma) \arrow[r, "\partial_{n-2}"] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r, "\partial_1"] & C_0(K_\sigma) \arrow[r, "\varepsilon"] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z} \arrow[d, phantom, "\shortparallel"] \\
C_n(\Sigma_\sigma) \arrow[r, "\partial_n'"] \arrow[d, phantom, "\shortparallel"] & C_{n-1}(\Sigma_\sigma) \arrow[r, "\partial_{n-1}'"] & C_{n-2}(\Sigma_\sigma) \arrow[r] & \cdots \arrow[r] & C_0(\Sigma_\sigma) \arrow[r, "\varepsilon'"] & \mathbb{Z} \\
0 & & & & &
\end{tikzcd}$$

The images and kernels of ∂_p and ∂_p' are the same for $p \leq n-2$. Hence $\tilde{H}_p(\Sigma_\sigma) = \tilde{H}_p(K_\sigma) = 0$ for $p < n-2$.

Consider $\tilde{H}_{n-1}(\Sigma_\sigma)$. Since $B_{n-1}(\Sigma_\sigma) = \operatorname{im} \partial_n' = 0$ we have

$$\tilde{H}_{n-1}(\Sigma_\sigma) = Z_{n-1}(\Sigma_\sigma).$$

Now,

$$Z_{n-1}(\Sigma_\sigma) = \ker \partial_{n-1}' = \ker \partial_{n-1} = \operatorname{im} \partial_n,$$

since $\tilde{H}_{n-1}(K_\sigma) = 0$.

Now $C_n(K_\sigma) = \langle\sigma\rangle \cong \mathbb{Z}$. Thus $\operatorname{im} \partial_n$ is generated by $\cancel{\partial}$ $\partial\sigma$. Thus $Z_{n-1}(\Sigma_\sigma)$ is generated by $\partial\sigma$ and is free by definition. Thus,

$$\tilde{H}_n(\Sigma_\sigma) = Z_n(\Sigma_\sigma) = \langle\partial\sigma\rangle \cong \mathbb{Z}.$$