Section 9: Relative Homology and Excision

Def Let K be a complex with K_0 a subcomplex.

Let $C_p(K, K_0) = C_p(K)/C_p(K_0)$.

We want to define a boundary map $\partial_p^{rel} : C_p(K, K_0) \to C_{p-1}(K, K_0)$.
Consider:

$$\begin{tikzcd} C_p(K) \arrow[r, "\partial_p"] \arrow[d, "\pi_p"] & C_{p-1}(K) \arrow[d, "\pi_{p-1}"] \\ C_p(K)/C_p(K_0) \arrow[r, "?", "\partial_p^{rel}"'] & C_{p-1}(K)/C_{p-1}(K_0) \end{tikzcd}$$

Let $\partial_p^{rel} = \pi_{p-1} \circ \partial_p \circ$ "π_p^{-1}". We must show it is well defined.

Let $[c_p] \in C_p(K)/C_p(K_0)$, i.e., $[c_p] = c_p + C_p(K_0)$.

Let d_p be any member of $C_p(K_0)$. So, $c_p + d_p \in [c_p]$,

i.e., $c_p + d_p \in \pi_p^{-1}([c_p])$.

Claim $\partial_p^{rel}([c_p])$ is independent of d_p.

Pf: $\partial_p(c_p + d_p) = \partial c_p + \partial d_p$, but $\partial d_p \in C_{p-1}(K_0)$.
Thus,

$$\pi_{p-1}(\partial c_p + \partial d_p) = \pi_{p-1}(\partial c_p) \in C_{p-1}(K)/C_{p-1}(K_0).$$

Next we check that $\partial_{p-1}^{rel}\,\partial_p^{rel} = 0$.

$$\partial_{p-1}^{rel}\,\partial_p^{rel}([c_p]) = \partial_{p-1}^{rel}([\partial_p c_p]) = [\partial_{p-1}\partial_p c_p] = [0] = 0.$$

Now we can define the <u>relative homology groups.</u>

<u>Def</u> $\quad Z_p(K, K_0) = \ker \partial_p^{rel}$

$B_p(K, K_0) = \operatorname{im} \partial_{p+1}^{rel}$

$H_p(K, K_0) = Z_p(K, K_0) / B_p(K, K_0)$.

<u>Thm</u> $\quad$ Let K be a complex and v a vertex in K. Then

$$H_n(K, v) = \tilde{H}_n(K) \quad , \forall n.$$

pf: $\quad$ #3, Section 9.

Example $K = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2$ $\qquad K_0 = \; v_0 \qquad v_2$

Find $H_0(K, K_0)$ and $H_1(K, K_0)$.

$$C_0(K, K_0) = \frac{C_0(K)}{C_0(K_0)} = \frac{\langle v_0, v_1, v_2 \rangle}{\langle v_0, v_2 \rangle} = \langle v_1 + C_0(K_0) \rangle = \langle [v_1] \rangle$$

$$C_1(K, K_0) = C_1(K) = \langle e_1, e_2 \rangle$$

H_0 $\quad \ker \partial_0^{rel} = C_0(K, K_0).$

$\operatorname{im} \partial_1^{rel} = ?$

$$\begin{aligned} \partial_1^{rel} e_1 &= \partial e_1 + C_0(K_0) \\ &= v_1 - v_0 + C_0(K_0) \\ &= v_1 + C_0(K_0) \\ &= [v_1] \\ \partial_1^{rel} e_2 &= \partial e_2 + C_0(K_0) \\ &= v_2 - v_1 + C_0(K_0) \\ &= -v_1 + C_0(K_0) \\ &= -[v_1] \end{aligned}$$

Thus $\operatorname{im} \partial_1^{rel} = \langle [v_1] \rangle = \ker \partial_0^{rel}$.

Thus,

$$H_0(K, K_0) = \frac{\langle [v_1] \rangle}{\langle [v_1] \rangle} = 0.$$

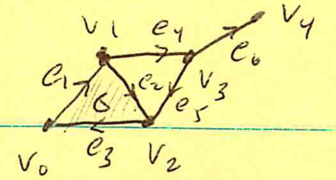
H_1 $\quad H_1(K, K_0) = \ker \partial_1^{rel} / \operatorname{im} \partial_2^{rel} = \ker \partial_1^{rel}.$

Set $\partial_1^{rel}(n_1 e_1 + n_2 e_2) = 0$.

Then $n_1 [v_1] - n_2 [v_1] = 0. \Rightarrow n_1 = n_2.$

Thus, $H_1(K, K_0) = \langle e_1 + e_2 \rangle \cong \mathbb{Z}.$

Discuss.

Example $K =$ 

$P = \{v_1, v_4\}$

$L = \{v_1, v_2, v_3, e_2, e_4, e_5\}$.

Find $H_0(K,P)$, $H_1(K,P)$, $H_0(K,P)$, $H_0(K,L)$.

$$C_0(K,P) = C_0(K)/C_0(P) = \frac{\langle v_0, v_1, v_2, v_3, v_4 \rangle}{\langle v_1, v_4 \rangle} = \langle [v_0], [v_0], [v_5] \rangle.$$

$$C_1(K,P) = C_1(K) = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle.$$

$H_0(K,P)$ $\quad \ker \partial_0^{rel} = C_0(K,P)$

$\operatorname{im} \partial_1^{rel} = ?$

$$\begin{aligned}
\partial_1^{rel} e_1 &= v_1 - v_0 + C_0(P) = -[v_0] \\
\partial_1^{rel} e_2 &= v_2 - v_1 + C_0(P) = [v_2] \\
\partial_1^{rel} e_3 &= v_0 - v_2 + C_0(P) = [v_0] - [v_2] \\
\partial_1^{rel} e_4 &= v_3 - v_1 + C_0(P) = [v_3] \\
\partial_1^{rel} e_5 &= v_2 - v_3 + C_0(P) = [v_2] - [v_5] \\
\partial_1^{rel} e_6 &= v_4 - v_3 + C_0(P) = -[v_3]
\end{aligned}$$

Thus, $\operatorname{im} \partial_1^{rel} = \langle [v_0], [v_2], [v_5] \rangle = \ker \partial_0^{rel}$

Hence $H_0(K,P) = 0$.

$H_1(K,P)$

Find $\ker \partial_1^{rel}$.

Set $\partial_1^{rel}(n_1e_1+n_2e_2+n_3e_3+n_4e_4+n_5e_5+n_6e_6)=0$.

Thus $-n_1[v_0]+n_2[v_2]+n_3[v_0]-n_3[v_2]+n_4[v_3]+n_5[v_2]-n_5[v_3]$
$-n_6[v_3]=0$

Or $(-n_1+n_3)[v_0]+(n_2-n_3+n_5)[v_2]+(n_4-n_5-n_6)[v_3]=0$.

$n_1=n_3$
$n_2=n_3-n_5$
$n_4=n_5+n_6$

$$\begin{bmatrix} n_1\\ n_2\\ n_3\\ n_4\\ n_5\\ n_6 \end{bmatrix} = \begin{bmatrix} 1\\ 1\\ 1\\ 0\\ 0\\ 0 \end{bmatrix} n_3 + \begin{bmatrix} 0\\ -1\\ 0\\ 1\\ 1\\ 0 \end{bmatrix} n_5 + \begin{bmatrix} 0\\ 0\\ 0\\ 1\\ 0\\ 1 \end{bmatrix} n_6$$

span null space

$Z_1(K,P)=\ker\partial_1^{rel}=\langle e_1+e_2+e_3,\ -e_2+e_4+e_5,\ e_4+e_6\rangle$

$B_1(K,P)=B_1(K)=\operatorname{im}\partial_2=\langle e_1+e_2+e_3\rangle$

Thus,

$$H_1(K,P)\cong\mathbb{Z}^2.$$

$$C_0(K,L) = C_0(K)/C_0(L) = \frac{\langle v_0, v_1, v_2, v_3, v_4\rangle}{\langle v_1, v_2, v_3\rangle} = \langle [v_0], [v_4]\rangle$$

$$C_1(K,L) = C_1(K)/C_1(L) = \frac{\langle e_1, e_2, e_3, e_4, e_5, e_6\rangle}{\langle e_2, e_4, e_5\rangle} = \langle [e_1], [e_3], [e_6]\rangle$$

(*)

$H_0(K,L)$

$$\ker \partial_0^{rel} = C_0(K,L) = \langle [v_0], [v_4]\rangle$$

$\mathrm{im}\, \partial_1^{rel} = ?$

$$\partial_1^{rel}[e_1] = -[v_0]$$
$$\partial_1^{rel}[e_3] = [v_0]$$
$$\partial_1^{rel}[e_6] = [v_4]$$

Thus $\mathrm{im}\, \partial_1^{rel} = \langle [v_0], [v_4]\rangle$.

Hence $H_0(K,L) = 0$.

$H_1(K,L)$

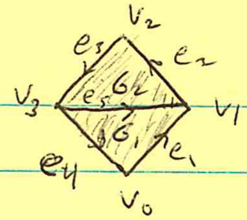
$$\ker \partial_1^{rel} = \langle [e_1 + e_3]\rangle$$

$\mathrm{im}\, \partial_2^{rel} = ?$ $\quad \partial_2^{rel}\sigma = [e_1 + e_2 + e_3] = [e_1 + e_3]$

Thus, $H_1(K,L) = 0$.

<u>Discuss.</u>

(*) $C_2(K,L) = C_2(K) = \langle \sigma \rangle$.

Example $K =$ 

$L = \{e_1, e_2, e_3, e_4, v_0, v_1, v_2, v_3\}$.

Find $H_0(K,L)$, $H_1(K,L)$ and $H_2(K,L)$.

$C_0(K,L) = 0$ $\quad C_1(K,L) = \langle [e_5] \rangle$

$$C_2(K,L) = C_2(K) = \langle \sigma_1, \sigma_2 \rangle.$$

H_0 $\quad H_0 = 0$

H_1 $\quad H_1 = \dfrac{\ker \partial_1^{rel}}{\operatorname{im} \partial_2^{rel}}$ $\qquad \partial_1^{rel}[e_5] = 0$

$\Rightarrow \ker \partial_1^{rel} = \langle [e_5] \rangle.$

$\operatorname{im} \partial_2^{rel} = ?$ $\qquad \partial_2^{rel} \sigma_1 = -[e_5]$

$\partial_2^{rel} \sigma_2 = [e_5].$

Thus $\operatorname{im} \partial_2^{rel} = \langle [e_5] \rangle$

$$H_1(K,L) = \frac{\langle [e_5] \rangle}{\langle [e_5] \rangle} = 0.$$

H_2. $\quad \operatorname{im} \partial_3^{rel} = 0$ so $H_2(K,L) = \ker \partial_2^{rel}$.

Set $\partial_2^{rel}(n_1 \sigma_1 + n_2 \sigma_2) = 0$.

Then $(-n_1 + n_2)[e_5] = 0 \Rightarrow n_1 = n_2$.

Thus $\ker \partial_2^{rel} = \langle \sigma_1 + \sigma_2 \rangle$, and

$$H_2(K,L) = \langle \sigma_1 + \sigma_2 \rangle \cong \mathbb{Z}.$$

Excision

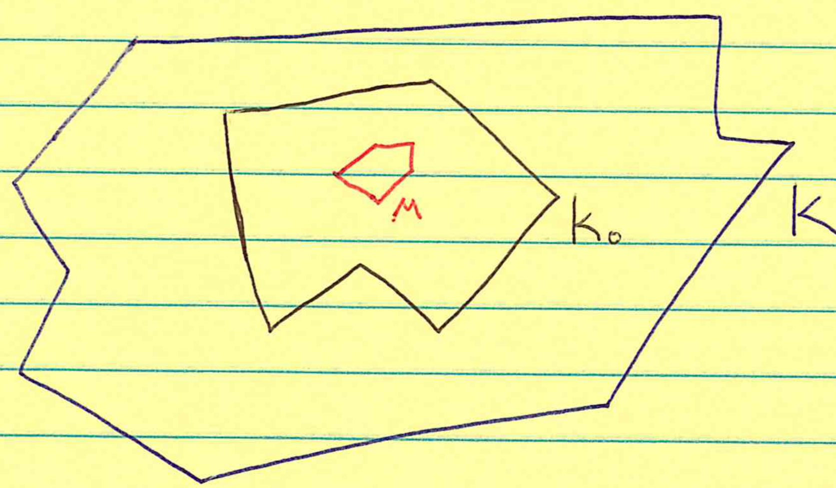
Idea Let K be a complex and K_0 a subcomplex. Compute $H_n(K, K_0)$. If we now punch a hole in K that is inside K_0, the rel. homology groups should not change.

Thm 9.1 Let M, K_0, L, L_0 be subcomplexes of K s.t.

$$|M| \subset |K_0|$$
$$|L| = |K| - \operatorname{int}|M|$$
$$|L_0| = |K_0| - \operatorname{int}|M|$$

Then

$$H_p(K, K_0) \cong H_p(L, L_0), \quad \forall p.$$



(In the textbook $U = \operatorname{int}|M|$ is used instead.)

Why this is important will be seen later.

The textbook gives a short sketchy "proof." I'll do the details here, but you may skip this on a first read. We need the following algebra theorem first.

Thm Consider the following diagram of abelian groups and homomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha_{p+2}"] & A_{p+1} \arrow[r, "\alpha_{p+1}"] \arrow[d, "\psi_{p+1}"'] & A_p \arrow[r, "\alpha_p"] \arrow[d, "\psi_p"'] & A_{p-1} \arrow[r, "\alpha_{p-1}"] \arrow[d, "\psi_{p-1}"'] & \cdots \\
\cdots \arrow[r, "\beta_{p+2}"'] & B_{p+1} \arrow[r, "\beta_{p+1}"'] & B_p \arrow[r, "\beta_p"'] & B_{p-1} \arrow[r, "\beta_{p-1}"'] & \cdots
\end{tikzcd}$$

Suppose $\alpha_p \circ \alpha_{p+1} = 0$ and $\beta_p \circ \beta_{p+1} = 0$ $\forall p$, and that ψ_p is an isomorphism $\forall p$. Assume each square commutes: $\psi_p \circ \alpha_{p+1} = \beta_{p+1} \circ \psi_{p+1}$, $\forall p$. Then,

$$\frac{\ker \alpha_p}{\operatorname{im} \alpha_{p+1}} \cong \frac{\ker \beta_p}{\operatorname{im} \beta_{p+1}}, \quad \forall p.$$

[In words: isomorphic chain-complexes have isomorphism homology groups.]

Pf Let $[a] \in \ker \alpha_p / \operatorname{im} \alpha_{p+1}$.

Define $\Psi_p : \frac{\ker \alpha_p}{\operatorname{im} \alpha_{p+1}} \to \frac{\ker \beta_p}{\operatorname{im} \beta_{p+1}}$ by $\Psi_p([a]) = [\psi_p(a)]$.

① Ψ_p is well-defined. Let $a' = a + q$, $q \in \operatorname{im} \alpha_{p+1}$.

Let $b' = \psi_p(a')$ and $b = \psi_p(a)$. Does $[b'] = [b]$?

Since $b' = \psi_p(a) + \psi_p(q) = b + \psi_p(q)$ we need

to show $\psi_p(q) \in \operatorname{im} \beta_{p+1}$.

Let $z \in A_{p+1}$ be s.t. $\alpha_{p+1}(z) = q$.

Now $\psi_p(q) = \psi_p(\alpha_{p+1}(z)) = \beta_{p+1}(\psi_{p+1}(z)) \in \operatorname{im} \beta_{p+1}$.

Thus $[b'] = [b]$.

② Ψ is onto.

Let $[b] \in \frac{\ker \beta_p}{\operatorname{im} \beta_{p+1}}$. Let $a = \psi_p^{-1}(b)$.

Then $\Psi_p([a]) = [\psi_p(a)] = [b]$.

③ Ψ_p is one-to-one.

Suppose $\Psi_p([a]) = id$.

Then $\psi_p(a) \in \operatorname{im} \beta_{p+1}$

Let $q \in B_{p+1}$ be s.t. $\beta_{p+1}(q) = \psi_p(a)$.

Let $z = \psi^{-1}_{p+1}(q)$. Then

$\psi_p(\alpha_{p+1}(z)) = \beta_{p+1}(\psi_{p+1}(z)) = \beta_{p+1}(q) = \psi_p(a)$.

Thus, $\alpha_{p+1}(z) = a$, so $a \in \operatorname{im} \alpha_{p+1}$.

Thus $[a] = id$.

Hence Ψ_p is one-to-one. $\blacksquare$

Pf of Excision Thm

Let $\phi_p = \pi \circ i$ where

$$C_p(L) \xhookrightarrow{i} C_p(K) \xrightarrow{\pi} C_p(K)/C_p(K_0).$$

We claim ϕ_p induces an isomorphism

$$\hat{\phi}_p : C_p(L)/C_p(L_0) \cong C_p(K)/C_p(K_0).$$

Claim ϕ_p is onto. Let $\mathcal{S} = \{[\sigma_i] \mid \sigma_i \text{ is a } p\text{-simplex in } K\}$. Here $[\sigma_i] = \sigma_i + C_p(K_0)$. Then $\mathcal{S}$ generates $C_p(K)/C_p(K_0)$. If $\sigma_i \notin K_0$, then $\sigma_i \in C_p(L)$, and $\phi(\sigma_i) = [\sigma_i]$. If $\sigma_i \in K_0$, then $[\sigma_i] = 0$. Thus, the set $\{\phi(\sigma_i) \mid \sigma_i \text{ is a } p\text{-simplex in } L\}$ also generates $C_p(K)/C_p(K_0)$. This proves the claim.

Claim $\ker \phi_p = C_p(L_0)$. Let $\sigma \in L$, be a p-simplex.

If $\sigma \in L_0$ then $\sigma \in K_0$ and $\phi(\sigma) = 0$.
If $\sum n_i \sigma_i \in C_p(L_0)$, then $\phi(\sum n_i \sigma_i) = \sum n_i 0 = 0$.
Thus, $C_p(L_0) \subset \ker \phi$.

If σ is not in L_0, then it is not in K_0.
Thus $\phi(\sigma) = [\sigma] \neq 0$. Thus nothing in $C_p(L) - C_p(L_0)$ (as sets) is in the kernel of ϕ. Hence, $\ker \phi = C_p(L_0)$.

Standard facts from algebra imply ϕ_p induces an isomorphism

$$\hat{\phi}_p \; C_p(L, L_0) \longrightarrow C_p(K, K_0), \quad \forall p.$$

Now consider

$$\begin{tikzcd}
\cdots \arrow[r] & C_p(L, L_0) \arrow[r, "\partial_p'^{\,rel}"] \arrow[d, "\hat{\phi}_p"] & C_{p-1}(L, L_0) \arrow[r] \arrow[d, "\hat{\phi}_{p-1}"] & \cdots \\
\arrow[r] & C_p(K, K_0) \arrow[r, "\partial_p^{rel}"] & C_{p-1}(K, K_0) \arrow[r] & \cdots
\end{tikzcd}$$

If we can show each such square commutes, it will follow that the homology groups are isomorphic. Watch.

$$\begin{tikzcd}
\arrow[r] & C_p(L, L_0) \arrow[r, "\pi_p^{-1}"'] \arrow[d, "\hat{\phi}_p"] \arrow[rrr, bend left, "\partial_p^{rel}"] & C_p(L) \arrow[r, "\partial_p"'] & C_{p-1}(L) \arrow[r, "\pi_{p-1}"'] & C_{p-1}(L, L_0) \arrow[r] \arrow[d, "\hat{\phi}_{p-1}"] & {} \\
\arrow[r] & C_p(K, K_0) \arrow[r, "\pi_p^{-1}"'] \arrow[rrr, bend right, "\partial_p^{rel}"'] & C_p(K) \arrow[r, "\partial_p"'] & C_{p-1}(L) \arrow[r, "\pi_{p-1}"'] & C_{p-1}(K, K_0) &
\end{tikzcd}$$

Let $[d_p]_L \in C_p(L, L_0)$ and suppose d_p is carried by $|L| - |L_0|$. Since $|L| - |L_0| = |K| - |K_0|$, we have $\hat{\phi}_p [d_p]_L = [d_p]_K$.

$$[d_p]_L = d_p + C_p(L_0) \longrightarrow d_p + l_p \rightarrow \partial d_p + \partial l_p \longrightarrow \partial d_p + C_{p-1}(L_0)$$

$$[d_p]_K = d_p + C_p(K_0) \rightarrow d_p + \underbrace{l_p' + m_p}_{C_p(L_0) \oplus C_p(M)} \longrightarrow \partial d_p + \partial(l_p' + m_p) \longrightarrow \partial d_p + C_{p-1}(K_0)$$

But, $\hat{\phi}_{p-1}\big(\partial d_p + C_{p-1}(L_0)\big) = \partial d_p + C_{p-1}(K_0)$. $\square$