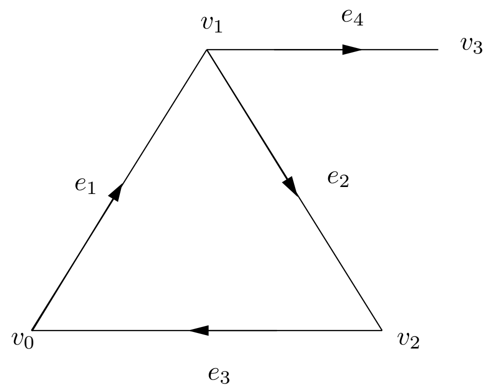
Homology Calculations

Example 1. Let K be the complex depicted below.



We have $C_0 = \langle v_0, v_1, v_2, v_3 \rangle \cong \mathbb{Z}^4$, and $C_1 = \langle e_1, e_2, e_3, e_4 \rangle \cong \mathbb{Z}^4$. All the other chain groups are trivial.

Find $\mathbf{H}_0(\mathbf{K})$. $Z_0 = \ker \partial_0 : C_0 \to 0 = 0$. Thus $Z_0 = C_0$. Now $B_0 = \text{im } \partial_1 : C_1 \to C_0$. Thus,

$$B_0 = \langle \partial_1 e_1, \partial_1 e_2, \partial_1 e_3, \partial_1 e_4 \rangle = \langle v_1 - v_0, v_2 - v_1, v_0 - v_2, v_3 - v_1 \rangle .$$

Using the standard bases for C_0 and C_1 the matrix for ∂_1 is

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, $H_0(K) = Z_0/B_0 \cong \dfrac{\mathbb{Z}^4}{A\mathbb{Z}^4}$. Since the Smith normal form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we get $H_0 \cong \mathbb{Z}$.

Find $\mathbf{H}_1(\mathbf{K})$. Since there are no 2-simplices $B_1 = 0$ and $H_1 = Z_1 = \ker \partial_1 : C_1 \to C_0$. By using the matrix A to represent ∂_1 the kernel of ∂_1 corresponds to the null space of A. We can find a basis for the null space:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, $H_1(K) \cong \mathbb{Z}$,

Below are the commands used in Maple.

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A:=Matrix([[-1,0,1,0],[1,-1,0,-1],[0,1,-1,0],[0,0,0,1]]);
SmithForm(A);
NullSpace(A);
```

The point of this example is that the computation of homology groups is completely reducible to linear algebra. This presents a sharp contrast with the homotopy groups.