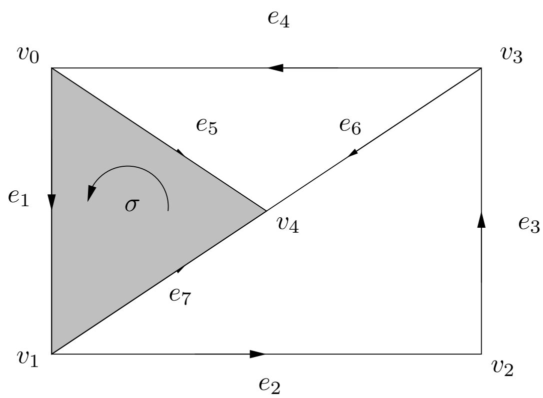
Homology Calculations

Example 2. Let K be the complex depicted below.



We have $C_0 = \langle v_0, v_1, v_2, v_3, v_4 \rangle \cong \mathbb{Z}^5$, $C_1 = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle \cong \mathbb{Z}^7$, and $C_2 = \langle \sigma \rangle \cong \mathbb{Z}$. All the other chain groups are trivial.

Find $\mathbf{H}_0$. $Z_0 = \ker \partial_0 : C_0 \to 0 = 0$. Thus $Z_0 = C_0$. Now $B_0 = \text{im } \partial_1 : C_1 \to C_0$. Thus, $B_0 = \langle \partial_1 e_1, \partial_1 e_2, \partial_1 e_3, \partial_1 e_4, \partial_1 e_5, \partial_1 e_6, \partial_1 e_7 \rangle =$
$\langle v_1 - v_0, v_2 - v_1, v_3 - v_2, v_0 - v_3, v_4 - v_0, v_4 - v_3, v_4 - v_1 \rangle$. Using the standard bases for C_0 and C_1 the matrix for ∂_1 is

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Thus $H_0 = Z_0/B_0 \cong \dfrac{\mathbb{Z}^5}{A\mathbb{Z}^7}$. Since the Smith normal form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get $H_0 \cong \mathbb{Z}$.

Find $\mathbf{H}_1$. $Z_1 = \ker \partial_1$. By visual inspection we can see three cycles that should generate Z_1. This gives $Z_1 = \langle e_1 + e_7 - e_5, e_5 - e_6 + e_4, e_2 + e_3 + e_6 - e_7 \rangle \cong \mathbb{Z}^3$. Clearly $B_1 = \langle \partial_2 \sigma \rangle = \langle e_1 + e_7 - e_5 \rangle$. Since our generator for B_1 is one of our generators for Z_1 we get $H_1 = Z_1/B_1 \cong \mathbb{Z}^2$.

But, what if Z_1 was not visually obvious? By using the matrix A to represent ∂_1 the kernel of ∂_1 corresponds to the null space of A. We can find a basis for the null space (this basis is not canonical):

$$\left\{\begin{bmatrix}2 & 1 & 1 & 1 & -1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 1 & 0 & -1 & 1 & 0\end{bmatrix}\right\}$$

I did not pick out the same cycles as before, but you can check that these two are equivalent. But, our generator for B_1 is no longer easily seen as a generator of Z_1. We just need to solve the systems of equations

$$\begin{bmatrix}2 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\\ -1 & 0 & 1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\\ 0\\ -1\\ 0\\ 1\end{bmatrix}$$

Thus $a = 1$, $b = -1$ and $c = 0$. In the new basis B_1 is generated by $\begin{bmatrix}1 & -1 & 0\end{bmatrix}$. If B_1 had more generators we could do this for each of them. Thus

$$H_1 = Z_1/B_1 \cong \frac{\mathbb{Z}^3}{\begin{bmatrix}1 & -1 & 0\end{bmatrix}\mathbb{Z}^3} \cong \frac{\mathbb{Z}^3}{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\mathbb{Z}^3} \cong \mathbb{Z}^3/\mathbb{Z} \cong \mathbb{Z}^2$$

Find $\mathbf{H_2}$. There are no 2-cycles so $H_2 = 0$.

For $n > 2$, $H_n = 0$.

The point of this example is that the computation of homology groups is completely reducible to linear algebra. This presents a sharp contrast with the homotopy groups.