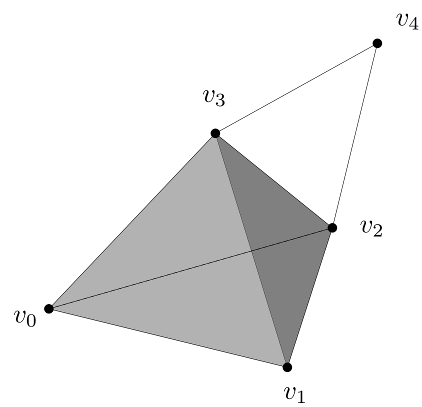
Homology Calculations

Example 3. Let K be the complex depicted below. We expect to find $H_0(K) \cong \mathbb{Z}$, $H_1(K) \cong \mathbb{Z}$, and $H_2(K) \cong \mathbb{Z}$.



We use the following labels.

$$\begin{array}{ll} \sigma_1 = [v_0, v_1, v_3] & e_1 = [v_0, v_1] \\ \sigma_2 = [v_1, v_2, v_3] & e_2 = [v_1, v_3] \\ \sigma_3 = [v_2, v_0, v_3] & e_3 = [v_3, v_0] \\ \sigma_4 = [v_0, v_1, v_2] & e_4 = [v_1, v_2] \\ & e_5 = [v_2, v_3] \\ & e_6 = [v_0, v_2] \\ & e_7 = [v_3, v_4] \\ & e_8 = [v_4, v_2] \end{array}$$

You should write these onto the figure.

We have $C_0 = \langle v_0, v_1, v_2, v_3, v_4 \rangle \cong \mathbb{Z}^5$, $C_1 = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \rangle \cong \mathbb{Z}^8$, and $C_2 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \rangle \cong \mathbb{Z}^4$.

We present a matrix function $A_1 : \mathbb{Z}^8 \to \mathbb{Z}^5$ that represents $\partial_1 : C_1 \to C_0$, and a matrix function $A_2 : \mathbb{Z}^4 \to \mathbb{Z}^8$ that represents $\partial_2 : C_2 \to C_1$.

$$A_1 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find $\mathbf{H}_0$. $H_0 = \dfrac{\ker \partial_0}{\text{im } \partial_1} = \dfrac{C_0}{\text{im } \partial_1} \cong \dfrac{\mathbb{Z}^5}{A_1 Z^8}$. As you can check the Smith normal form of A_1 is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $H_0 \cong \mathbb{Z}$.

Find $\mathbf{H_1}$. $H_1 = \dfrac{\ker \partial_1}{\operatorname{im} \partial_2}$. The kernel of ∂_1 is isomorphic to the kernel of A_1, which can be computed using the NullSpace command in Maple. For a basis of the NullSpace we get

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

To get a basis for the image of ∂_2 we use the ColumnSpace command to get

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

It is not obvious how to compute $\dfrac{\text{span } \mathcal{B}_1}{\text{span } \mathcal{B}_2}$ since the members of $\mathcal{B}_2$ are not expressed in terms of the members of $\mathcal{B}_1$. We rectify this by solving for the linear combinations of members of $\mathcal{B}_1$ that give the members of $\mathcal{B}_2$, that is we solve $[\mathcal{B}_1]X = [\mathcal{B}_2]$. The Maple command LinearSolve can do this. We get

$$X = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Check this. The Smith normal form of X is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$H_1 \cong \frac{\mathbb{Z}^4}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{Z}^3} \cong \mathbb{Z}.$$

Find $\mathbf{H_2}$. $H_2 = \dfrac{\ker \partial_2}{\text{im } \partial_3} = \ker \partial_2$. We get that the null space of A_2 has basis $\{[-1, -1, -1, 1]^T\}$. Thus, $H_2 \cong \mathbb{Z}$.