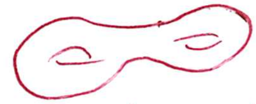
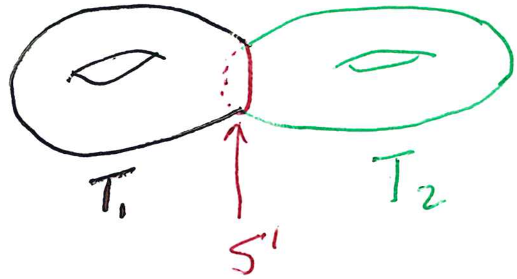
Example

Find homology gps of $T^2 \# T^2$



D = double torus



$D = T_1 \cup T_2$

$S^1 = T_1 \cap T_2$

A punctured torus is homotopic to a wedge of two circles. Thus, we know the homology gps of T_1 and T_2.

The M-V seq gives

$$\begin{array}{ccccccccccccc}
H_2(T_1) \oplus H_2(T_2) & \xrightarrow{\alpha} & H_2(D) & \xrightarrow{\beta} & H_1(S^1) & \xrightarrow{\gamma} & H_1(T_1) \oplus H_1(T_2) & \xrightarrow{\delta} & H_1(D) & \xrightarrow{\rho} & \tilde{H}_1(S^1) & \to & \\
 & & \| & & \cong & & \cong \quad \cong & & \| & & \| & & \\
0 \oplus 0 & & ? & & \mathbb{Z} & & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & & ? & & 0 & &
\end{array}$$

$\operatorname{im} \alpha = 0 \Rightarrow \ker \beta = 0 \Rightarrow \beta$ is 1-to-1.

Let $z \in H_1(S^1)$ be a generator. Since z bounds a 2-chain in T_1 and in T_2 $\gamma(z) = (0,0)$. Thus, $\operatorname{im} \gamma = 0$. $\ker \gamma = H_1(S^1)$

$\Rightarrow \beta$ is onto $\Rightarrow \beta$ is a iso. $\Rightarrow H_2(D) \cong \mathbb{Z}$.

$\ker \rho = H_1(D) \Rightarrow \operatorname{im} \delta = H_1(D)$. So δ is onto.

As noted, $\operatorname{im} \gamma = 0 \Rightarrow \ker \delta = 0$. So is an iso.

Thus, $H_1(D) \cong \mathbb{Z}^4$.

$$H_p(D) \cong \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}^4 & p = 1 \\ \mathbb{Z} & p = 2 \end{cases}$$

Example Find homology gps of $T^3 = S^1 \times S^1 \times S^1$.

We know $H_p(T^2) = \begin{cases} \mathbb{Z} & p=2, \\ \mathbb{Z}^2 & p=1, \\ \mathbb{Z} & p=0. \end{cases}$ We know $H_0(T^3) \cong \mathbb{Z}$.

Use Exercise 6 for the rest.

$$H_1(T^2 \times S^1) \cong H_0(T^2) \oplus H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}^2 = \mathbb{Z}^3$$

$$H_2(T^2 \times S^1) = H_1(T^2) \oplus H_2(T^2) \cong \mathbb{Z}^2 \oplus \mathbb{Z} = \mathbb{Z}^3.$$

$$H_3(T^2 \times S^1) = H_2(T^2) \oplus H_3(T^2) \cong \mathbb{Z} \oplus 0 = \mathbb{Z}.$$

$$H_p(T^3) \cong \begin{cases} \mathbb{Z} & p=0 \\ \mathbb{Z}^3 & p=1 \\ \mathbb{Z}^3 & p=2 \\ \mathbb{Z} & p=3 \end{cases}$$

Example Let $X = T^3 - \mathring{B}^3$, this is we remove an open 3-ball from T^3. Find homology gps.

Now $T^3 = X \cup B^3$ and $S^2 = X \cap B^3$.

$$\begin{array}{ccccccc} H_1(S^2) & \to & H_1(X) \oplus H_1(B^3) & \to & H_1(T^3) & \to & \tilde{H}_0(S^2) \\ \| & & \| \quad\quad \| & & & & \| \\ 0 & & ? \;\oplus\; 0 & & & & 0 \end{array}$$

is exact.

Thus, $H_1(X) \cong H_1(T^3) \cong \mathbb{Z}^3$.

$$\begin{array}{ccccccc} H_2(S^2) & \xrightarrow{\alpha} & H_2(X) \oplus H_2(B^3) & \longrightarrow & H_2(T^3) & \to & H_1(S^2) \\ \text{\rotatebox{90}{$\cong$}} & & \| \quad\quad \| & & \text{\rotatebox{90}{$\cong$}} & & \| \\ \mathbb{Z} & & ? \;\oplus\; 0 & & \mathbb{Z}^3 & & 0 \end{array}$$

But, $\operatorname{im}\alpha = (0,0)$ since S^2 the boundary of X and B^3.

Thus, $H_2(X) \cong H_2(T^3) \cong \mathbb{Z}^3$.

Finally,

$$\begin{array}{ccccccccc} H_3(S^2) & \xrightarrow{\alpha} & H_3(X) \oplus H_3(B^3) & \xrightarrow{\beta} & H_3(T^3) & \xrightarrow{\cancel{\text{onto}}} & H_2(S^2) & \to & (0,0) \in H_2(X) \oplus H_2(B) \\ \| & & \| \quad\quad \| & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \\ 0 & \to & ? \;\oplus\; 0 & \to & \mathbb{Z} & \xrightarrow[\text{onto}]{} & \mathbb{Z} & \to & (0,0) \end{array}$$

Only exist two onto ~~maps~~ hom's from $\mathbb{Z}$ to $\mathbb{Z}$, id and $\times(-1)$.
Both have $\ker = 0$. Thus $\operatorname{im}\beta = 0$. But β is 1-to-1.

Thus $H_3(X) = 0$.

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$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}^3 & p = 1 \\ \mathbb{Z}^3 & p = 2 \\ 0 & p = 3 \end{cases}$$

Example: Find homology group of $T^3 \# T^3$.

Note: The definition of the connected sum $(\#)$ of two 3-manifolds is: remove an open 3-ball from each and identify the two resulting boundary spheres.

Now, we can write $T^3 \# T^3 = (T_1^3 - \text{int } B_1^3) \underset{\partial}{\cup} (T_2^3 - \text{int } B_2^3)$.

and note that $(T_1^3 - \text{int } B_1^3) \cap (T_2^3 - \text{int } B_2^3) = S^2$.

Let $D = T_1^3 \# T_2^3$, $X = T_1^3 - \text{int}(B_1^3)$, $Y = T_2^3 - \text{int } B_2^3$.

$$\begin{array}{ccccccc} H_1(S^2) & \to & H_1(X) \oplus H_1(Y) & \to & H_1(D) & \to & \tilde{H}_0(S^2) \\ \| & & \cong \quad\quad \cong & & \| & & \| \\ 0 & & \mathbb{Z}^3 \oplus \mathbb{Z}^3 & & ? & & 0 \end{array}$$

is exact.

Thus $H_1(D) \cong H_1(X) \oplus H_1(Y) \cong \mathbb{Z}^6$.

$$\begin{array}{ccccccc} H_2(S^2) & \xrightarrow{\alpha} & H_2(X) \oplus H_2(Y) & \to & H_2(D) & \to & H_1(S^2) \\ \cong & & \cong \quad\quad \cong & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z}^3 \oplus \mathbb{Z}^3 & & ? & & 0 \end{array}$$

$\text{im } \alpha = (0,0)$ since S^2 is the boundary of X and of Y.

Thus, $H_2(D) \cong \mathbb{Z}^6$.

$$\begin{array}{ccccccc} H_3(X) \oplus H_3(Y) & \to & H_3(D) & \to & H_2(S^2) & \xrightarrow{\alpha} & (0,0) \\ \| \quad\quad \| & & & & \cong & & \\ 0 \oplus 0 & \to & ? & \to & \mathbb{Z} & \to & 0 \end{array}$$

is exact

Thus, $H_3(D) \cong H_2(S^2) \cong \mathbb{Z}$.

Thus,

$$H_p(T^3 \# T^3) \cong \begin{cases} \mathbb{Z} & p=0 \\ \mathbb{Z}^6 & p=1 \\ \mathbb{Z}^6 & p=2 \\ \mathbb{Z} & p=3 \end{cases}$$