

Lecture Notes for Section 12

Part I. Simplicial maps and induced homomorphisms.

Part II. Contiguous simplicial maps and chain homotopies.

Part III. Extend this to relative homology groups [read on your own].

Part I

DEFINITION. (From Section 2) Let K and L be complexes. Let f be a function from the vertices of K to the vertices of L . While f need not be one-to-one or onto we assume that the vertices of each simplex in K are taken onto the vertices of some simplex of L so that we can extend f linearly to a continuous function $f : |K| \rightarrow |L|$. We abuse notation and write $f : K \rightarrow L$ for the *simplicial map* from K to L determined by the original vertex map.

DEFINITION. Let $f : K \rightarrow L$ be a simplicial map. We define the *induced homomorphism* on the chain groups

$$f_{\#} : C_p(K) \rightarrow C_p(L)$$

as follows. Let $\sigma = [v_0, \dots, v_p]$ be a p -simplex of K . Then $\{f(v_0), \dots, f(v_p)\}$ spans a p' -simplex of L where $p' \leq p$. Let

$$f_{\#}(\sigma) = \begin{cases} [f(v_0), \dots, f(v_p)] & \text{if } p' = p \\ 0 & \text{if } p' < p. \end{cases}$$

We can extend $f_{\#}$ to a homomorphism since $f_{\#}(-\sigma) = -f_{\#}(\sigma)$ is clear.

LEMMA (12.1). $f_{\#}$ commutes with ∂ .

Proof. See textbook.

COROLLARY. $f_{\#}$ induces a homomorphism on homology groups,

$$f_* : H_p(K) \rightarrow H_p(L).$$

Proof. Let $c \in H_p(K)$. Let z and z' be representative p -cycles of c , that is $c = z + B_p(K) = z' + B_p(K)$ for $z, z' \in Z(p(K))$. We will show that $f_{\#}(z) \sim f_{\#}(z')$. This means we can define $f_*(c) = f_{\#}(z)$ for any $z \in c$ without ambiguity.

(1) We claim $f_{\#}(z) \in Z_p(L)$. *Proof.* $\partial f_{\#}(z) = f_{\#}\partial(z) = f_{\#}(0) = 0$. Likewise $f_{\#}(z') \in Z_p(L)$.

(2) We claim $f_{\#}(z) \sim f_{\#}(z')$. Let $b = z - z'$. Since $z \sim z'$ we know $b \in B_p(K)$. Hence $\exists d \in C_{p+1}(K)$ with $\partial d = b$. Now

$$f_{\#}(b) = f_{\#}(\partial d) = \partial f_{\#}(d) \in B_p(L).$$

Thus, $f_{\#}(z) - f_{\#}(z') = f_{\#}(b) \in B_p(L)$ proving the claim.

Thus, we can define $f_*(c) = f_{\#}(z) + B_p(L)$ for any choice of $z \in c$ without ambiguity. \square

THEOREM (12.2). Let $*(f : K \rightarrow L) = \{f_* : H_p(K) \rightarrow H_p(L) \mid \forall p\}$. Then $*$ is a functor.

Outline of Proof. (a) Let $\text{id} : K \rightarrow K$ be the identity simplicial map. Then one checks that $\text{id}_* : H_p(K) \rightarrow H_p(K)$ is the identity isomorphism $\forall p$.

(b) Let $f : K \rightarrow L$ and $g : L \rightarrow M$ be simplicial maps. Then one checks that $(g \circ f)_* = g_* \circ f_*$. \square

FACT. This all works for reduced homology groups. See textbook.

Part II.

DEFINITION. Let $f, g : K \rightarrow L$ be simplicial maps. We say f and g are *contiguous* if $\forall \sigma = [v_0, \dots, v_p] \in K$ the set

$$\{f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)\}$$

spans a simplex in L .

REMARK. Contiguous is a standard English word that means next to.

Example. In the figure below define $f : K \rightarrow L$ by

$$f(v_0) = w_0 \quad f(v_1) = w_1 \quad f(v_2) = w_2$$

and $g : K \rightarrow L$ by

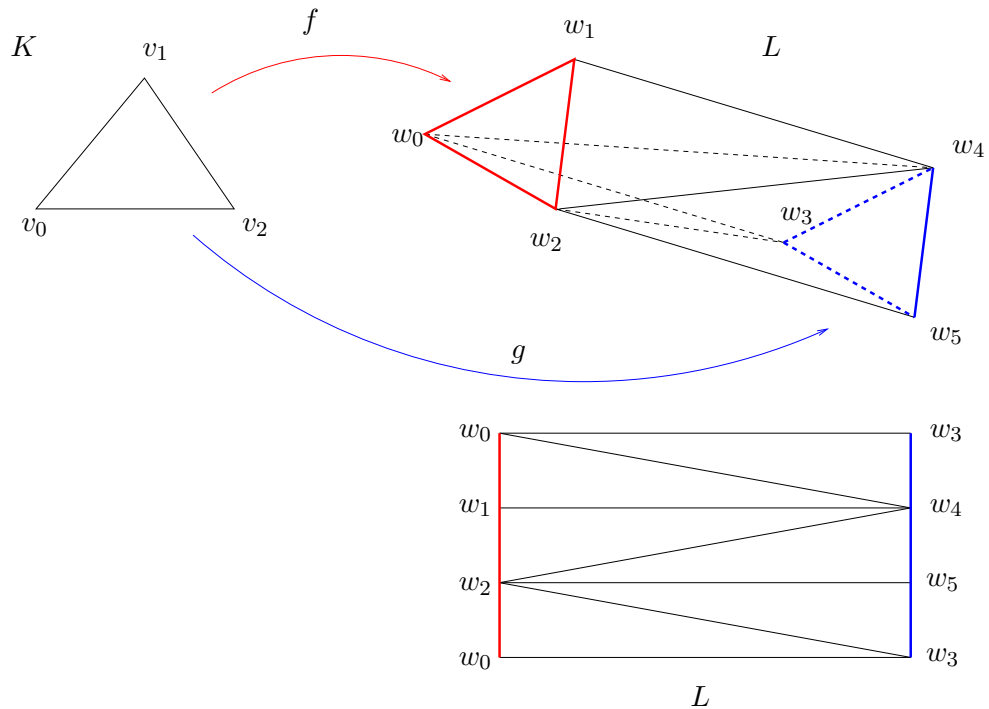
$$g(v_0) = w_3 \quad g(v_1) = w_4 \quad g(v_2) = w_5.$$

For each v_i the set $\{f(v_i), g(v_i)\}$ spans an edge of L . But for $\sigma = [v_0, v_1]$ we have the set

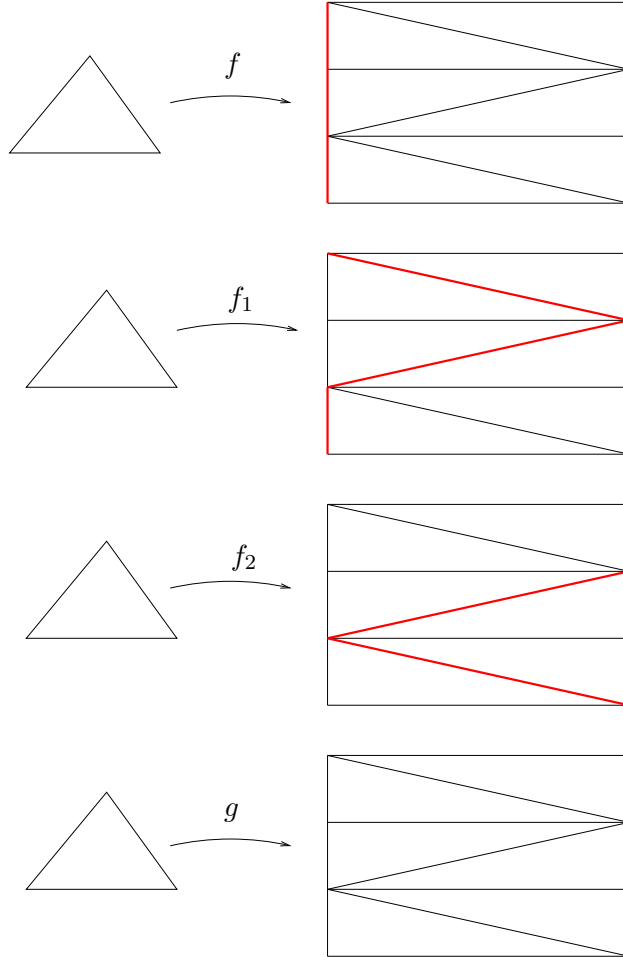
$$\{f(v_0), f(v_1), g(v_0), g(v_1)\} = \{w_0, w_1, w_3, w_4\}$$

and these do not span a simplex of L .

Thus, f and g are not contiguous.



But, consider the following simplicial maps of K into L .



Now f is contiguous to f_1 which is contiguous to f_2 which is contiguous to g .

DEFINITION. Whenever this happens we will say that f and g are *eventually contiguous*.

REMARK. Think of this as a discrete/combinatorial analog of homotopy.

Notice that in this example $f_* = g_*$. We will show that whenever f and g are eventually contiguous this happens.

The next definition will seem unnatural at first.

DEFINITION. Let $f, g : K \rightarrow L$ be simplicial maps. Suppose $\forall p \exists$ a homomorphism $D : C_p(K) \rightarrow C_{p+1}(L)$ s.t.

$$\partial D + D\partial = g_{\#} - f_{\#}.$$

Such a D is called a *chain homotopy* between f and g . When this happens we say f and g are *chain homotopic*.

THEOREM (12.4). If f and g are chain homotopic then $f_* = g_*$.

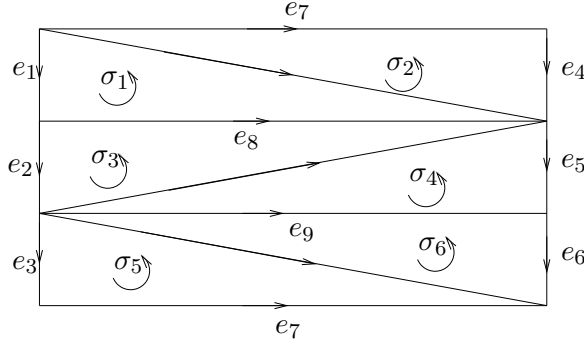
Proof. Let $z \in Z_p(K)$. Then

$$g_{\#}(z) - f_{\#}(z) = \partial Dz + D\partial z = \partial Dz \in B_p(L).$$

Thus, $g_{\#}(z) \sim f_{\#}(z) \forall z \in Z_p(K)$, so $g_* = f_*$. \square

Return to Example.

We shall construct D . Really all we need is to have $D : Z_p(K) \rightarrow C_{p+1}(L)$ be s.t. $\partial D(z) = g_{\#}(z) - f_{\#}(z)$. But this is hard to do because we would need to choose a basis for $Z_p(K)$ and define D on these basis cycles and then extend to $Z_p(K)$. Instead we define D on all of $C_p(K)$ by defining D on p -simplices. This is easier, but the “cost” is the $D\partial$ term in the definition. But, $D\partial$ is always 0 on cycles and so causes no harm.



For $p = 0$, let $D(v_i) =$ the edge $[f_{\#}(v_i), g_{\#}(v_i)] \in C_1(L)$.

For $p = 1$, let $D([v_0, v_1]) = -(\sigma_1 + \sigma_2)$ in $C_2(L)$,

$$D([v_1, v_2]) = -(\sigma_3 + \sigma_4), \text{ and}$$

$$D([v_2, v_0]) = -(\sigma_5 + \sigma_6).$$

Check for $p = 0$: Let $c \in C_0(K)$, $c = \sum n_i v_i$. Then

$$\begin{aligned} \partial Dc &= \sum n_i \partial Dv_i = \sum n_i (g_{\#}(v_i) - f_{\#}(v_i)) \\ &= g_{\#}(c) - f_{\#}(c) = g_{\#}(c) - f_{\#}(c) - D\partial c, \end{aligned}$$

since $D\partial c = 0$.

Check for $p = 1$: Let $c = n_1[v_0, v_1] + n_2[v_1, v_2] + n_3[v_2, v_0] \in C_1(K)$. Then

$$\begin{aligned} \partial Dc &= -n_1\partial(\sigma_1 + \sigma_2) - n_2\partial(\sigma_3 + \sigma_4) - n_3\partial(\sigma_5 + \sigma_6) \\ &= -n_1(e_1 + e_8 - e_7 - e_4) - n_2(e_2 - e_8 + e_9 - e_5) - n_3(e_3 - e_9 + e_7 - e_6) \\ &= -(n_1e_1 + n_2e_2 + n_3e_3) + (n_1e_4 + n_2e_5 + n_3e_6) + (n_2 - n_1)e_8 + (n_3 - n_2)e_9 + (n_1 - n_3)e_7. \end{aligned}$$

Now,

$$-f_{\#}c = -(n_1e_1 + n_2e_2 + n_3e_3) \quad g_{\#}c = n_1e_4 + n_2e_5 + n_3e_6,$$

and

$$\begin{aligned} D\partial c &= D(n_1(v_1 - v_0) + n_2(v_2 - v_1) + n_3(v_0 - v_2)) \\ &= D((n_3 - n_1)v_0 + (n_1 - n_2)v_1 + (n_2 - n_3)v_2) \\ &= (n_3 - n_1)e_7 + (n_1 - n_2)e_8 + (n_2 - n_3)e_9. \end{aligned}$$

Thus,

$$\partial Dc + D\partial c = g_{\#}c - f_{\#}c.$$

In general, the construction of D can be quite ad hoc.

THEOREM (12.5). If $f, g : K \rightarrow L$ are eventually contiguous then \exists a chain homotopy between f and g , and hence $f_* = g_*$.

Proof. It is enough to prove this for f and g contiguous.

Let $\sigma = [v_0, \dots, v_p] \in K$. Let $L(\sigma)$ = the subcomplex of L whose vertex set is

$$\{f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)\};$$

and $L(\sigma)$ contains all faces formed from these.

We have the following facts.

- (1) $L(\sigma)$ is not empty and it is acyclic.
- (2) If s is a face of σ , then $L(s) \subset L(\sigma)$.
- (3) $\forall \sigma \in K$, the chains $f_{\#}(\sigma)$ and $g_{\#}(\sigma)$ are carried by $L(\sigma)$.

We must show a chain-homotopy D exists.

Let $p = 0$. \exists a 1-chain c in $L(v)$ from $f_{\#}(v)$ to $g_{\#}(v)$ for each vertex v of K since f and g are contiguous. Define $D(v) = c$. Then

$$\partial Dv + D\partial v = \partial c + 0 = g_{\#}(v) - f_{\#}(v).$$

Note that $D(v)$ is carried by $L(v)$.

Now, assume D is defined for all dimensions $< p$ and is such that \forall simplices s of dimension $< p$ we have that $D(s)$ is carried by $L(s)$ and

$$\partial Ds + D\partial s = g_{\#}(s) - f_{\#}(s).$$

Let σ be a p -simplex of K . Let

$$c = g_{\#}(\sigma) - f_{\#}(\sigma) - D\partial c.$$

Claim 1: c is a cycle. (See textbook.)

Claim 2: c is carried by $L(\sigma)$. (See textbook.)

Since $L(\sigma)$ is acyclic we know c is a boundary of some $p+1$ -chain in $L(\sigma)$. Let d be just such a $p+1$ -chain. Define $D\sigma = d$. Then

$$\partial D\sigma = \partial d = c = g_{\#}(\sigma) - f_{\#}(\sigma) - D\partial\sigma.$$

Thus,

$$\partial D\sigma + D\partial\sigma = g_{\#}(\sigma) - f_{\#}(\sigma).$$

We can extend D to p -chains. □

Part III

All this can be extended to relative homology groups. Read this on your own in the textbook.