Ch 3

The simplicial homology groups are actually only one of many "homology theories." A little later we will introduce singular homology theory. These theories are organized by 8 axioms. A central concept needed for the is the idea of long exact sequences of abelian gps.

These are defined in §23, and in §24 the purely algebraic Zig-Zag Lemma is proved. This is used to develop two long exact seq's of relative homology groups: The l.e.s of a pair and the Mayer-Vietoris Sequence. (§25)

As applications we compute the simp. hom. gps of the spheres, S^n, and products of spheres $S^n \times S^m$.

§26 gives the axioms. §27 proves these for simp hom., but we have done most of this it turns out, so I'll let you read it on your own. §28 define catagorries and functors - we've talked about these before, so you can skim this section.

§ 23

Def Consider a sequence of abelian gps and homomorphisms

$$\cdots \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \xrightarrow{\phi_4} \cdots$$

(it may be finite, infinite or bi-infinite). It is <u>exact</u> at n if image ϕ_{n-1} = kernel ϕ_n, exactly equal.

If it is exact at all n, it is called an <u>exact seq</u>.

If it is bi-infinite it is called a <u>long exact seq</u>.

If we have

$$0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0$$

we call it a <u>short exact seq</u>. Notice that this is equivalent to specifying that ϕ be injective and ψ be surjective.

A s.e.s <u>splits</u> if $A_2 = \phi(A_1) \oplus B$, for some gp B. It is easy to prove that this is eq to $\exists$ iso $\theta : A_2 \to A_1 \oplus A_3$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "\phi"] \arrow[d, "id"] & A_2 \arrow[r, "\psi"] \arrow[d, "\theta"] & A_3 \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A_1 \oplus A_3 \arrow[r] & A_3 \arrow[r] & 0
\end{tikzcd}$$

commutes. (see Thm 23.1; study it.)

Q: What can we say if $0 \to A \overset{f}{\hookrightarrow} B \to 0$ is exact?

There are several important exact seq's that often come up. Here are two

① Let K be a simplicial complex with subcomplex K_0. Let $i: K_0 \hookrightarrow K$ be inclusion and $\pi: (K, \emptyset) \to (K, K_0)$ be inclusion of pairs. Then, there exists a hom's ∂_*, (defined later) s.t.

$$\cdots \xrightarrow{\partial_*} H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K) \xrightarrow{i_*} \cdots$$

is a l.e.s, call the l.e.s of the pair (K, K_0).

② Let K be a simp. com. with subcom's K_0, K_1, s.t. $K = K_0 \cup K_1$. Let $A = K_0 \cap K_1$. Then $\exists$ hom's s.t.

$$\cdots \to H_p(A) \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(A) \to \cdots$$

is exact. It is call the Mayer-Victoris Seq of (K, K_0, K_1)

We will see later how these are used to compute various hom. gps. The MVS plays a role similar to the Seifert/van Kampen Thm from 530.

Both are derived from the Zig-Zag Lemma. Hold on to your seats!

§ 24 The Zig-Zag Lemma.

Def Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be chain complexes. Then

$$0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$$

is a <u>short exact seq of chain complexes</u> if $\forall p$

$$0 \to C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p \to 0$$

is exact.

We can now ~~state~~ state and prove the ZZ lemma. The text also ~~[illegible]~~ discusses some ~~[illegible]~~ other result, such as the Steenrod five-lemma and the exact seq of a triple (X, A, B) $(B \subset A \subset X)$, in exercise 1, and the serpent lemma in exercise 2. Study these.

The Zig-Zag Lemma

Thm Suppose one is given three chain complexes

$$\mathcal{C} = \{C_p, \partial_p^C\}_{p=-\infty}^{\infty}, \quad \mathcal{D} = \{D_p, \partial_p^D\}, \quad \mathcal{E} = \{E_p, \partial_p^E\}$$

and chain maps ϕ and ψ s.t. the sequence

$$0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$$

is exact. Then $\exists$ a long exact homology seq

$$\cdots \to H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \to \cdots$$

Proof Consider the commutative diagram

$$\begin{tikzcd}
 & {} \arrow[d] & & & \\
0 \arrow[r] & C_{p+1} \arrow[r, "\phi_{p+1}"] \arrow[d, "\partial_{p+1}^C"'] & D_{p+1} \arrow[r, "\psi_{p+1}"] \arrow[d, "\partial^D"'] & E_{p+1} \arrow[r] \arrow[d, "\partial^E"'] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\phi"] \arrow[d, "\partial"'] & D_p \arrow[r, "\psi"] \arrow[d, "\partial"'] & E_p \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "\phi"] \arrow[d] & D_{p-1} \arrow[r, "\psi"] \arrow[d] & E_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r] & D_{p-2} \arrow[r] & E_{p-2} \arrow[r] & 0
\end{tikzcd}$$

Note that

$$\ker \phi_p = \operatorname{Im} 0 = 0 \Rightarrow \phi_p \text{ is 1-to-1}, \ \forall p$$

and $\operatorname{Im} \psi_p = \operatorname{Ker} 0 = E_p \Rightarrow \psi_p$ is onto $\forall p$.

We will often leave off homomorphism subscripts. You can fill them in.

Step 1 First we define $\partial_*: H_p(\mathcal{E}) \to H_{p-1}(\mathcal{C})$.

Let $\alpha \in H_p(\mathcal{E})$. Let $e_p \in \operatorname{Ker} \partial_p^E \subset E_p$ be a cycle s.t. $\alpha = [e_p]$. Since ψ is onto we can choose $d_p \in D_p$ s.t. $\psi(d_p) = e_p$. The element $\partial^D d_p \in D_{p-1}$ lies in $\ker \psi_{p-1}$ since,

$$\psi(\partial^D d_p) = \partial^E \psi(d_p) = \partial^E e_p = 0, \quad (e_p \text{ is a cycle}).$$

~~Thus~~ $\exists\ c_{p-1} \in C_{p-1}$ s.t. $\phi(c_{p-1}) = \partial^D d_p$ since $\ker \psi = \operatorname{im} \phi$. Thus c_{p-1} is unique since ϕ is 1-to-1. Furthermore c_{p-1} is a cycle since

$$\phi(\partial^C c_{p-1}) = \partial^D \phi(c_{p-1}) = \partial^D \partial^D d_p = 0,$$

implies $\partial^C c_{p-1} = 0$ because ϕ is 1-to-1.

We define $\partial_* \alpha = [c_{p-1}] \in H_{p-1}(\mathcal{C})$.

Step 2 We show ∂_* is well defined and a homomorphism.

Let e_p and e_p' be cycles in $\alpha \in H_p(\mathcal{E})$.

Then $e_p - e_p' = \partial^E e_{p+1}$ for some $e_{p+1} \in E_{p+1}$.

Choose d_p and d_p' so that $\psi(d_p) = e_p$ and $\psi(d_p') = e_p'$.

Then choose c_{p-1} and c_{p-1}' s.t. $\phi(c_{p-1}) = \partial^D d_p$ and

$\phi(c_{p-1}') = \partial^D d_p'$. Choose e_{p+1} s.t. $\psi(d_{p+1}) = e_{p+1}$.

Now we have

$\psi(d_p - d_p' - \partial^D d_{p+1}) = e_p - e_p' - \partial^E \psi(d_{p+1}) = e_p - e_p' - \partial^E e_{p+1} = 0.$

Since $d_p - d_p' - \partial^D d_{p+1} \in \ker \psi$ it is in $\operatorname{Im} \phi$.

Choose c_p s.t. $\phi(c_p) = d_p - d_p' - \partial^D d_{p+1}$.

Then $\phi(\partial^C c_p) = \partial^D \phi(c_p) = \partial^D(\qquad) =$

$= \partial^D(d_p - d_p') = \phi(c_{p-1} - c_{p-1}')$.

Since ϕ is 1-to-1, $\partial^C c_p = c_{p-1} - c_{p-1}'$.

Thus $\partial_* \alpha = [c_p] = [c_p']$ is unambiguous.

To show ∂_* is a homomorphism let e_p and e_p' be any two cycles in E_p, with d_p, d_p', c_{p-1} and c_{p-1}' defined as above. We need to show

$$\partial_*([e_p]+[e_p']) = [c_{p-1}] + [c_{p-1}'].$$

Note that $\psi(d_p+d_p') = e_p+e_p'$ and

$\phi(c_{p-1}+c_{p-1}') = \partial^D(d_p+d_p')$ since ψ, ϕ + ∂^D are homomorphisms. Therefore, by def., $\partial_*[e_p+e_p'] = [c_{p-1}+c_{p-1}']$.

Since homology eq. classes respect ~~the~~ chain gp additions we get

$$\partial_*([e_p]+[e_p']) = [c_{p-1}] + [c_{p-1}']$$

as desired.

<u>Step 3</u> We prove exactness at $H_p(\mathcal{D})$.

(i) $\psi_* \phi_* = (\psi\phi)_* = 0_* = 0 \Rightarrow \operatorname{Im} \phi_* \subset \ker \psi_*$.

(ii) Conversely, let $\gamma = [d_p] \in \ker \psi_*$.

Since $\psi_*[d_p] = 0$, $\psi(d_p) = \partial^E e_{p+1}$ for some $e_{p+1} \in E_{p+1}$. Choose d_{p+1} s.t. $\psi(d_{p+1}) = e_{p+1}$.

Then

$$\psi(d_p - \partial^D d_{p+1}) = \psi(d_p) - \partial^E \psi(d_{p+1}) = \psi(d_p) - \partial^E e_{p+1} = 0.$$

Hence $d_p - \partial^D d_{p+1} \in \ker \psi = \operatorname{Im} \phi$ so $\exists\, c_p \in C_p$ s.t. $d_p - \partial^D d_{p+1} = \phi(c_p)$. Now c_p is a cycle since,

$$\phi(\partial^C c_p) = \partial^D \phi(c_p) = \partial^D d_p - \partial^D \partial^D d_{p+1} = 0 - 0,$$

and ϕ is one-to-one. Finally,

$$\phi_*[c_p] = [\phi(c_p)] = [d_p - \partial^D d_{p+1}] = [d_p],$$

so $[d_p] \in \operatorname{Im} \phi_*$ as desired.

Step 4 We prove exactness at $H_p(\mathcal{E})$.

Let $\alpha = [e_p] \in H_p(\mathcal{E})$. Suppose $\alpha \in \operatorname{Im}\psi_*$.

Let $d_p \in D_p$ be s.t. $\psi(d_p) = e_p$ as usual.

But since $\alpha \in \operatorname{Im}\psi_*$ we can take d_p to be a cycle. Let $c_{p-1} \in C_{p-1}$ be s.t. $\phi(c_{p-1}) = \partial^D d_p$ as usual.

But since d_p is a cycle we have $\phi(c_{p-1}) = 0$.

Since ϕ is 1-to-1, $c_{p-1} = 0$. Thus $\partial_* \alpha = [c_{p-1}] = 0$

and so $\alpha \in \ker \partial_*$. $\operatorname{Im}\psi_* \subset \ker \partial_*$.

Now suppose $\partial_* \alpha = 0$. Let $\partial_* \alpha = [c_{p-1}]$.

Then $c_{p-1} = \partial^C c_p$ for some $c_p \in C_p$. We assert that $d_p - \phi(c_p)$ is a cycle and $\alpha = \psi_*([d_p - \phi(c_p)])$. Thus $\alpha \in \operatorname{Im}\psi_*$.

By direct calculation:

$$\partial^D(d_p - \phi(c_p)) = \partial^D d_p - \phi\,\partial^C(c_p) = \partial^D d_p - \phi(c_{p-1}) = 0$$

$$\psi_*[d_p - \phi(c_p)] = [\psi(d_p) - \cancelto{0}{\psi\phi}(c_p)] = [e_p] = \alpha.$$

Thus $\ker \partial_* = \operatorname{Im}\psi_*$.

Step 5 We prove exactness at $H_{p-1}(\mathcal{C})$.

Let $\beta \in H_{p-1}(C)$. Suppose $\beta \in \operatorname{Im} \partial_*$.

Now $\beta = [c_{p-1}]$ and here $\phi(c_{p-1}) = \partial^D d_p$ for some $d_p \in D_p$, because β is in the image of ∂_*.

Then $\phi_*(\beta) = [\phi(c_{p-1})] = [\partial^D d_p] = 0$.

($\partial^D d_p$ is a boundary.)

Conversely, suppose $\phi_*(\beta) = 0$. Let $\beta = [c_{p-1}]$.

Then $[\phi(c_{p-1})] = 0$, $\Rightarrow \phi(c_{p-1}) = \partial^D d_p$ for some d_p. Let $e_p = \psi(d_p)$. Then e_p is a cycle since

$$\partial^E e_p = \psi(\partial^D d_p) = \psi\phi(c_{p-1}) = 0.$$

By def. $\beta = \partial_*[e_p]$. Thus $\beta \in \operatorname{Im} \partial_*$, and we are done. $\square$

Thm 23.3 (The exact homology seq of a pair.)

Let K be a complex, with K_0 a subcomplex. Then $\exists$ a l.e.s.

$$\cdots \to H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \to \cdots$$

where $i: K_0 \to K$ and $\pi: (K, \emptyset) \to (K, K_0)$ are inclusions.

Pf Check that $0 \to C_p(K_0) \xrightarrow{i_\#} C_p(K) \xrightarrow{\pi_\#} C_p(K, K_0) \to 0$ is exact $\forall p$.

Exactness at $C_p(K_0)$ follows since $i_\#$ is 1-to-1.

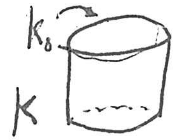
Now $\pi_\#$ is defined to take each element to its equivalence class. Thus it is onto and we have exactness at $C_p(K, K_0)$.

Let $c_p \in C_p(K_0)$. Then its support is in K_0. Now $i_\#(c_p) = c_p$ in $C_p(K)$. Since the support of c_p is in K_0, $\pi_\#(c_p) = 0$. And if $\pi_\#(c_p) = 0$, then the supp of c_p was in K_0 and thus in the image of $i_\#$.

Apply the Zig-Zag Lemma! ∎

Can be done in reduced homology as well. See text, bottom of page 138.

Example 3 "a"



$K = S^1 \times I$

$K_0 = S^1 \times \{1\}$,

$$\begin{array}{ccccccccccccccccc}
0 \to & H_2(K_0) & \to & H_2(K) & \to & H_2(K,K_0) & \to & H_1(K_0) & \xrightarrow{\cong} & H_1(K) & \to & H_1(K,K_0) & \to & H_0(K_0) & \to & H_0(K) & \to & H_0(K,K_0) \to 0 \\
& \| & & \| & & \| & & \cong & & \cong & & \| & & \| & & \| & & \| \\
& 0 & & 0 & & ? & & \mathbb{Z} & & \mathbb{Z} & & ? & & \mathbb{Z} & & \mathbb{Z} & & 0
\end{array}$$

kill with reduced homology

$$0 \to H_2(K,K_0) \xrightarrow{\alpha} \underset{\mathbb{Z}}{\overset{\|}{H_1(K_0)}} \xrightarrow{\cong} \underset{\mathbb{Z}}{\overset{\|}{H_1(K)}} \xrightarrow{\beta} H_1(K,K_0) \to 0$$

$\ker \alpha = 0$ α is 1-1 $\operatorname{im} \alpha = \ker(\cong) = 0$

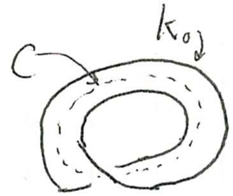
$\Rightarrow H_2(K,K_0) = 0$

β is onto

$\ker \beta = \operatorname{im} \cong = \mathbb{Z}$

$\Rightarrow H_1(K,K_0) = 0$.

Example 3 "b"

K 

$$0 \to \underset{\overset{\shortparallel}{0}}{H_2(K_0)} \to \underset{\overset{\shortparallel}{0}}{H_2(K)} \to \underset{\overset{\shortparallel}{?}}{H_2(K,K_0)} \xrightarrow{\alpha} \underset{\overset{\shortparallel}{\mathbb{Z}}}{H_1(K_0)} \xrightarrow{\times 2} \overset{\langle c\rangle \cong \mathbb{Z}}{\overset{\shortparallel}{H_1(K)}} \xrightarrow{\beta} \underset{\overset{\shortparallel}{?}}{H_1(K,K_0)} \to \underset{\overset{\shortparallel}{0}}{\tilde{H}_0(K_0)} \to \underset{\overset{\shortparallel}{0}}{\tilde{H}_0(K)} \to \underset{\overset{\shortparallel}{0}}{H_0(K,K_0)} \to 0$$

$\ker \alpha = 0 \Rightarrow \alpha$ 1-1

$\operatorname{im} \alpha = \ker \times 2 = 0$

$\Rightarrow H_2(K,K_0) = 0$

β is onto

$\ker \beta = \operatorname{im} \times 2 = \text{evens}$

$\Rightarrow H_1(K,K_0) \cong \mathbb{Z}_2$

Hmk §23 #6