

## Section 25 Mayer-Vietoris Sequence

Thm 25.1 Let  $K$  be a complex with subcomplexes  $K_0, K_1$ , s.t.  
 $K = K_0 \cup K_1$ ;  $A = K_0 \cap K_1$ .  $\exists$  hom's making

$$\dots \rightarrow H_p(A) \rightarrow H_p(K_0) \oplus H_p(K_1) \rightarrow H_p(K) \rightarrow H_{p-1}(A) \rightarrow \dots$$

a long exact seq. It is called the Mayer-Vietoris Seq. of  $K_0, K_1, K$ . If  $A \neq \emptyset$ ,  $\exists$  a similar l.e.s for reduced homology.

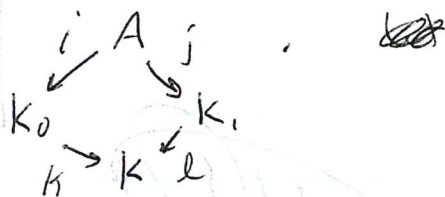
Pf

We define the chain complex  $C(K_0) \oplus C(K_1)$  to be  $\{C_p(K_0) \oplus C_p(K_1), d'_p(d, e) = (d_p^{K_0} d, d_p^{K_1} e)\}_{p \geq 0}$ . It is clear  $d'_p d'_{p+1} = 0$ , so this is a chain complex.

We construct a short exact seq of chain complexes

$$0 \rightarrow C(A) \xrightarrow{\phi} C(K_0) \oplus C(K_1) \xrightarrow{\psi} C(K) \rightarrow 0$$

where  $\phi$  and  $\psi$  are defined as follows. Consider the inclusion maps



$\phi$  Let  $c \in C_p(A)$ . Let  $\phi_p(c) = (i_{\#}(c), -j_{\#}(c)) \in C_p(K_0) \oplus C_p(K_1)$

$\psi$  Let  $d \in C_p(K_0), e \in C_p(K_1)$ . Let  $\psi(d, e) = k_{\#}(d) + l_{\#}(e) \in C_p(K)$ .

We check for exactness

$C_p(A)$ : Since  $i$  and  $j$  are inclusions,  $i_{\#}$  and  $j_{\#}$  have  $\ker = 0$ .  
Thus  $\phi(c) = (0, 0)$  iff  $c = 0$ . Thus  $\ker \phi = \text{im } 0 = 0$ .

$C_p(k)$ : We need to show  $\text{im } \psi = C_p(k)$ . Let  $d \in C_p(k)$ .  
Let  $d_0$  be the part of  $d$  carried by  $k_0$ . Then  $d_0 \in C_p(k_0)$ .  
Since  $k_0 \oplus k_1 = k$ ,  $d - d_0$  is carried by  $k_1$ , and so  $d - d_0 \in C_p(k_1)$ . We have

$$\psi(d_0, d - d_0) = k_{\#}(d_0) + l_{\#}(d - d_0) = d_0 + d - d_0 = d.$$

$C_p(k_0) \oplus C_p(k_1)$ : Let  $c \in C_p(A)$ .  $\psi \phi(c) = \psi(i_{\#}(c), -j_{\#}(c))$   
 $= \psi(c, -c) = k_{\#}(c) + l_{\#}(-c) = c - c = 0$ .

Thus  $\text{im } \phi \subset \ker \psi$ .

Now we show the other inclusion. Let  $(d, e) \in \ker \psi$ .  
Thus  $e = -d$ . Hence  $\phi(d) = (d, -d) = (d, e)$ . Thus  $(d, e) \in \text{im } \phi$ .

We apply the Zig-Zag Lemma to get the l.e.s

$$\dots \rightarrow H_p(A) \rightarrow H_p(\mathcal{C}(k_0) \oplus \mathcal{C}(k_1)) \rightarrow H_p(k) \rightarrow H_{p-1}(A) \rightarrow \dots$$

Finally, elementary alg gives  $H_p(k_0) \oplus H_p(k_1) \cong H_p(\mathcal{C}(k_0) \oplus \mathcal{C}(k_1))$ .

To see this note

$$\frac{\ker d_p^1}{\operatorname{im} d_{p+1}^1} = \frac{\ker d_p^{k_0} \oplus \ker d_p^{k_1}}{\operatorname{im} d_{p+1}^{k_0} \oplus \operatorname{im} d_{p+1}^{k_1}} \cong \frac{\ker d^{k_0}}{\operatorname{im} d^{k_0}} \oplus \frac{\ker d^{k_1}}{\operatorname{im} d^{k_1}},$$

since  $\operatorname{im} d^{k_0} \triangleleft \ker d^{k_0}$  and  $\operatorname{im} d^{k_1} \triangleleft \ker d^{k_1}$ . \*

□

\* Just check cosets. Let  $H_i \subset G_i$ ,  $i=1,2$ .

$$\text{Let } x \in \frac{G_1 \oplus G_2}{H_1 \oplus H_2}. \text{ Then } x = (g_1, g_2) + H_1 \oplus H_2 \\ = (g_1 + H_1, g_2 + H_2)$$

$$\text{Let } x \in \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}. \text{ Then } x = (g_1 + H_1, g_2 + H_2).$$