Section 25 Mayer-Vietoris Sequence

Thm 25.1 Let K be a complex with subcomplexes K_0, K_1, s.t. $K = K_0 \cup K_1$; $A = K_1 \cap K_0$. $\exists$ hom's making

$$\cdots \to H_p(A) \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(A) \to \cdots$$

a long exact seq. It is called the Mayer-Vietoris Seq. of K, K_0, K_1. If $A \neq \emptyset$, $\exists$ a similar l.e.s for reduced homology.

Pf We define the chain complex $\mathcal{C}(K_0) \oplus \mathcal{C}(K_1)$ to be $\{ C_p(K_0) \oplus C_p(K_1),\ \partial_p'(d,e) = (\partial_p^{K_0} d, \partial_p^{K_1} e) \}_{\forall p}$. It is clear $\partial' \partial' = 0$, so this is a chain complex.

We construct a short exact seq of chain complexes

$$0 \to \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K_0) \oplus \mathcal{C}(K_1) \xrightarrow{\psi} \mathcal{C}(K) \to 0$$

where ϕ and ψ are defined as follows. Consider the inclusion maps

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ K_0 \arrow[dr, "k"'] & & K_1 \arrow[dl, "l"] \\ & K & \end{tikzcd}$$

ϕ Let $c \in C_p(A)$. Let $\phi_p(c) = (i_\#(c_p), -j_\#(c)) \in C_p(K_0) \oplus C_p(K_1)$

ψ Let $d \in C_p(K_0)$, $e \in C_p(K_1)$. Let $\psi(d,e) = k_\#(d) + l_\#(e) \in C_p(K)$.

We check for exactness

$C_p(A)$: Since i and j are inclusions, $i_\#$ and $j_\#$ have $\ker = 0$.
Thus $\phi(c) = (0,0)$ iff $c = 0$. Thus $\ker \phi = \operatorname{im} 0 = 0$.

$C_p(K)$: We need to show $\operatorname{im} \psi = C_p(K)$. Let $d \in C_p(K)$.
Let d_0 be the part of d carried by K_0. Then $d_0 \in C_p(K_0)$.
Since $K_0 \cup K_1 = K$, $d - d_0$ is carried by K_1, and so $d - d_0 \in C_p(K_1)$. We have

$$\psi(d_0, d - d_0) = k_\#(d_0) + l_\#(d - d_0) = d_0 + d - d_0 = d.$$

$C_p(K_0) \oplus C_p(K_1)$: Let $c \in C_p(A)$. $\psi\phi(c) = \psi(i_\#(c), -j_\#(c))$
$= \psi(c, -c) = k_\#(c) + l_\#(-c) = c - c = 0.$
Thus $\operatorname{im} \phi \subset \ker \psi$.

Now we show the other inclusion. Let $(d, e) \in \ker \psi$.
Thus $e = -d$. Hence $\phi(d) = (d, -d) = (d, e)$. Thus $(d, e) \in \operatorname{im} \phi$.

We apply the Zig-Zag lemma to get the l.e.s

$$\cdots \to H_p(A) \to H_p(\mathcal{C}(K_0) \oplus \mathcal{C}(K_1)) \to H_p(K) \to H_{p-1}(A) \to \cdots$$

Finally, elementary alg gives $H_p(K_0) \oplus H_p(K_1) \cong H_p(\mathcal{C}(K_0) \oplus \mathcal{C}(K_1))$.
To see this note

$$\frac{\ker \partial_p^{!}}{\operatorname{im} \partial_{p+1}^{!}} = \frac{\ker \partial_p^{K_0} \oplus \ker \partial^{K_1}}{\operatorname{im} \partial_{p+1}^{K_0} \oplus \operatorname{im} \partial^{K_1}} \cong \frac{\ker \partial^{K_0}}{\operatorname{im} \partial^{K_0}} \oplus \frac{\ker \partial^{K_1}}{\operatorname{im} \partial^{K_1}},$$

since $\operatorname{im} \partial^{K_0} \triangleleft \ker \partial^{K_0}$ and $\operatorname{im} \partial^{K_1} \triangleleft \ker \partial^{K_1}$. *

$\square$

* Just check cosets. Let $H_i < G_i$, $i = 1, 2$.

Let $x \in \dfrac{G_1 \oplus G_2}{H_1 \oplus H_2}$. Then $x = (g_1, g_2) + H_1 \oplus H_2$
$= (g_1 + H_1, g_2 + H_2)$

Let $x \in \dfrac{G_1}{H_1} \oplus \dfrac{G_2}{H_2}$. Then $x = (g_1 + H_1, g_2 + H_2)$.