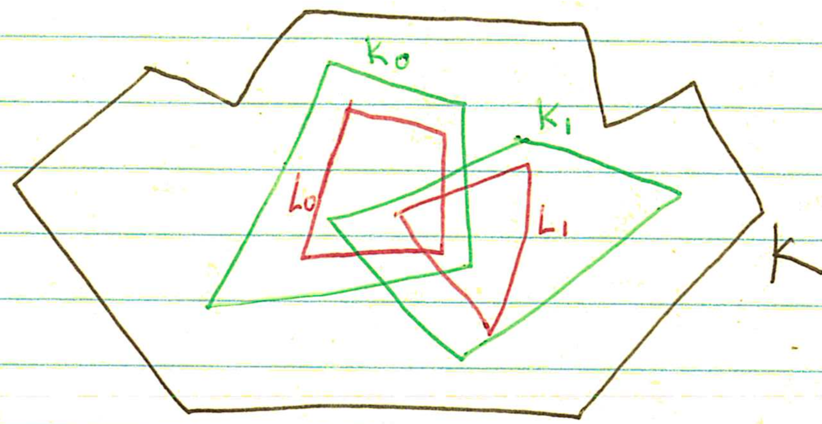
Section 25

#6

The goal of this problem is to compute the homology groups of $S^n \times S^m$. We will need to make use of Exercise #2.

#2 Let K_0 and K_1 be subcomplexes of K.
Let L_0 and L_1 be subcomplexes of K_0 and K_1 resp.



Then $\exists$ a long exact sequence:

$$\cdots \to H_n(K_0 \cap K_1, L_0 \cap L_1) \to H_n(K_0, L_0) \oplus H_n(K_1, L_1) \to H_n(K_0 \cup K_1, L_0 \cup L_1)$$
$$\to H_{n-1}(K_0 \cap K_1, L_0 \cap L_1) \to$$

It is called the relative Mayer-Vietoris seq.

#6a Let $p \in S^n$. Show that $H_q(X \times S^n, X \times p) \cong H_{q-n}(X)$.

Hint Write $S^n = U^n \cup L^n$ where U^n is the upper hemisphere and L^n is the lower hemisphere (wrt one axis) and proceed by induction on n.

Pf We do the $n=0$ case separately. Now $S^0 =$ two pts $= \{N, S\}$. Let $p = S$. Now $X \times \{N,S\}$ is just two copies of X. Then

$$H_q(X \times \{N,S\}, X \times S) \cong H_q(X \times N) \cong H_q(X) = H_{q-0}(X).$$

Let $n \geq 1$. We set up some machinery before we do the induction. We use the hint: $S^n = U^n \cup L^n$. Notice $U^n \cap L^n = S^{n-1}$, the "equator" of S^n. We pick $p \in S^{n-1}$.

To apply Exercise #2, use $K_0 = X \times U^n$, $K_1 = X \times L^n$ and $L_0 = L_1 = X \times p$. Then $K_0 \cap K_1 = X \times S^{n-1}$ and $L_0 \cup L_1 = L_0 \cap L_1 = X \times p$.

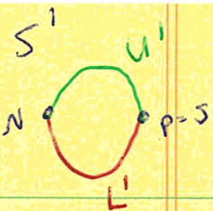
Notice, $X \times U^n$ and $X \times L^n$ are homotopic to $X \times p$ since U^n and L^n are disks that contain p. Thus $H_q(X \times U^n, X \times p) = 0$ and $H_q(X \times L^n, X \times p) = 0$.

The relative MV seq is

$$\cdots \to \cancelto{0}{H_q(X \times U^n, X \times p)} \oplus \cancelto{0}{H_q(X \times L^n, X \times p)} \to H_q(X \times S^n, X \times p) \to H_{q-1}(X \times S^{n-1}, X \times p)$$

$$\to \cancelto{0}{H_{q-1}(X \times U^n, X \times p)} \oplus \cancelto{0}{H_q(U \times L^n, X \times p)} \to \cdots$$

Thus, $H_q(X \times S^n, X \times p) \cong H_{q-1}(X \times S^{n-1}, X \times p)$ (*)



Now suppose $n=1$. Then by $(\star)$ we have $(p=S)$

$$H_q(X\times S^1, X\times p) \cong H_{q-1}(X\times S^0, X\times p) \cong H_{q-1}(X), \ \forall q.$$

Let $n \geq 1$ be fixed and assume we know

$$H_q(X\times S^n, X\times p) \cong H_{q-n}(X) \quad \forall q.$$

Then

$$\begin{aligned} H_q(X\times S^{n+1}, X\times p) &\overset{\star}{\cong} H_{q-1}(X\times S^n, X\times p) \\ &\cong H_{q-1-n}(X) \\ &= H_{q-(n+1)}(X). \end{aligned}$$

This proves (a).

6 (b) Show that if $p \in Y$, then the homology exact seq of $(X \times Y, X \times p)$ breaks up into short exact sequences that split.

What does this mean? Suppose:

$$\xrightarrow{\delta} A_{n+1} \xrightarrow{\phi} B_{n+1} \xrightarrow{\psi} C_{n+1} \xrightarrow{\delta} A_n \xrightarrow{\phi} B_n \xrightarrow{\psi} C_n \xrightarrow{\delta} A_{n-1} \xrightarrow{\phi} B_{n-1} \xrightarrow{\psi} C_{n-1} \xrightarrow{\delta} \cdots$$

is exact. If we could show that ϕ was always injective and ψ was always surjective, then for each n we would have that

$$0 \longrightarrow A_n \xrightarrow{\phi} B_n \xrightarrow{\psi} C_n \longrightarrow 0$$

is a short exact seq. When this happens we say the original l.e.s. "breaks up into short ex. seqs."

Notice, if we know each ϕ is inj. that is enough!
$\ker \phi = 0 \Rightarrow \operatorname{im} \delta = 0 \Rightarrow \ker \delta = C_n \Rightarrow \operatorname{im} \psi = C_n$, so ψ is onto. (see item (4) on pg 130.)

Now, consider the l.e.s. of $(X \times Y, X \times p)$

$$\to H_n(X \times p) \xrightarrow{i_*} H_n(X \times Y) \xrightarrow{\pi_*} H_n(X \times Y, X \times p) \xrightarrow{\partial_*} H_{n-1}(X \times p) \to \cdots$$

~~The inclusion map i X~~ Notice $r: (X \times Y) \to X \times p$ is a retraction. By Exercise 1(b) in §18 (pg 103) i_* is injective. Thus the les breaks up:

$$0 \longrightarrow H_p(X\times p) \xrightarrow{i_*} H_n(X\times Y) \xrightarrow{\pi_*} H_n(X\times Y, X\times p) \rightarrow 0$$

is exact for each n.

That ~~it~~ splits, i.e. $H_n(X\times Y) \cong H_n(X\times p) \oplus H_p(X\times Y, X\times p)$, follows from Exercise 6(a) Section 24, (pg 141) since, again, $X\times p$ is a retract of $X\times Y$. (The proof uses Thm 23.1 (pg 131-2).).

6(c) Prove that $H_q(X \times S^n) \cong H_{q-n}(X) \oplus H_q(X)$.

From 6(b) we have

$$H_q(X \times S^n) \cong H_q(X \times p) \oplus H_q(X \times S^n, X \times p)$$
$$\cong \quad\quad \cong \text{(a)}$$
$$H_q(X) \oplus H_{q-n}(X).$$

Thus

(d) Compute the homology of $S^n \times S^m$.

We will do some examples first.

$S^0 \times S^0$, $S^0 \times S^1$, $S^1 \times S^0$, $S^1 \times S^1$ are easy.

$H_0(S^0 \times S^0) = \mathbb{Z}^4$

$H_0(S^1 \times S^1) = \mathbb{Z}$

$H_0(S^0 \times S^1) = \mathbb{Z}^2$

$H_1(S^0 \times S^1) = \mathbb{Z}^2$ $H_1(S^1 \times S^1) = \mathbb{Z}^2$.

$H_2(S^1 \times S^1) = \mathbb{Z}$.

Let's try $S^3 \times S^5$. For $q < 0$ or > 8 we get zero. So we check $q = 0, 1, \cdots 8$.

$H_0(S^3 \times S^5) \cong \mathbb{Z}$

$H_1(S^3 \times S^5) \cong H_{-4}(S^3) \oplus H_1(S^3) = 0 \oplus 0 = 0$

$H_2(S^3 \times S^5) \cong H_{-3}(S^3) \oplus H_2(S^3) = 0 \oplus 0 = 0$

$H_3(S^3 \times S^5) = H_{-2}(S^3) \oplus H_3(S^3) \cong \mathbb{Z}$

$H_4(S^3 \times S^5) = H_{-1}(S^3) \oplus H_4(S^3) = 0$

$H_5(S^3 \times S^5) = H_0(S^3) \oplus H_5(S^3) = \mathbb{Z} \oplus 0 = \mathbb{Z}$

$H_6(S^3 \times S^5) = H_1(S^3) \oplus H_6(S^3) = 0$

$H_7(S^3 \times S^5) = H_2(S^3) \oplus H_7(S^3) = 0$

$H_8(S^3 \times S^5) = H_3(S^3) \oplus H_8(S^3) = \mathbb{Z}$.

Conjecture

$$\underset{n < m}{H_q(S^n \times S^m)} = \begin{cases} \mathbb{Z} & q = 0, n, m, n+m \\ 0 & \text{otherwise} \end{cases}$$

$$H_q(S^n \times S^n) = \begin{cases} \mathbb{Z} & q = 0, \cancel{n+m}\ 2n. \\ \mathbb{Z}^2 & q = n \\ 0 & \text{otherwise}. \end{cases}$$

You finish.

Try $\cancel{S^2 \times S^4 \times S^7 \times S^7}$