§73 **Čech Cohomology**

Def A directed set is a set J with a relation $\leq$ s.t.

$\alpha \leq \alpha$, $\forall \alpha \in J$ $\quad$ $\alpha \leq \beta, \beta \leq \gamma \Rightarrow \alpha \leq \gamma$, $\forall \alpha, \beta, \gamma \in J$ $\quad$ $\forall \alpha, \beta \in J$, $\exists \delta \in J$ s.t. $\alpha \leq \delta$ and $\beta \leq \delta$.

Ex The integers. All subsets of a set.

Def A direct system of abelian groups is a collection of abelian groups indexed by a directed set J, $\{G_\alpha\}_{\alpha \in J}$, and a collection of homomorphisms indexed by $J \times J$ with $\alpha \leq \beta$, $\{ f_{\alpha\beta} : G_\alpha \to G_\beta \mid \alpha \leq \beta \}$, s.t.

$$f_{\alpha\alpha} = id_{G_\alpha}, \text{ and}$$

$$\alpha \leq \beta \leq \gamma \Rightarrow f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}.$$

$$\begin{tikzcd} G_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[rr, bend right, "f_{\alpha\gamma}"'] & G_\beta \arrow[r, "f_{\beta\gamma}"] & G_\gamma \end{tikzcd}$$

Ex Let $J = \mathbb{Z}^+$, $G_i = \mathbb{Z}$ and $f_{ij} = \times 2^{j-i}$. ($f_{i,i+1} = \times 2$)

$$\cdots \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots$$

Def Given a direct system of abelian groups $\{G_\alpha, f_{\alpha\beta}\}_{\alpha\le\beta}$ we define a new abelian group called the <u>direct limit</u>,

$$\varinjlim_{\alpha\in J} G_\alpha$$

as follows. Let H = the disjoint union of the G_α.
We define an equivalence relation on H:

Let $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$ (so both are in H).

Then $g_\alpha \sim g_\beta$ if $\exists\, \delta \in J$ s.t. $\alpha \le \delta$, $\beta \le \delta$ and

$$f_{\alpha\delta}(g_\alpha) = f_{\beta\delta}(g_\beta).$$

Then $\varinjlim_{\alpha\in J} G_\alpha = H/\sim$.

We define addition in H by

$$[g_\alpha] + [g_\beta] = [f_{\alpha\delta}(g_\alpha) + f_{\beta\delta}(g_\beta)].$$

One can check this gives an abelian group.

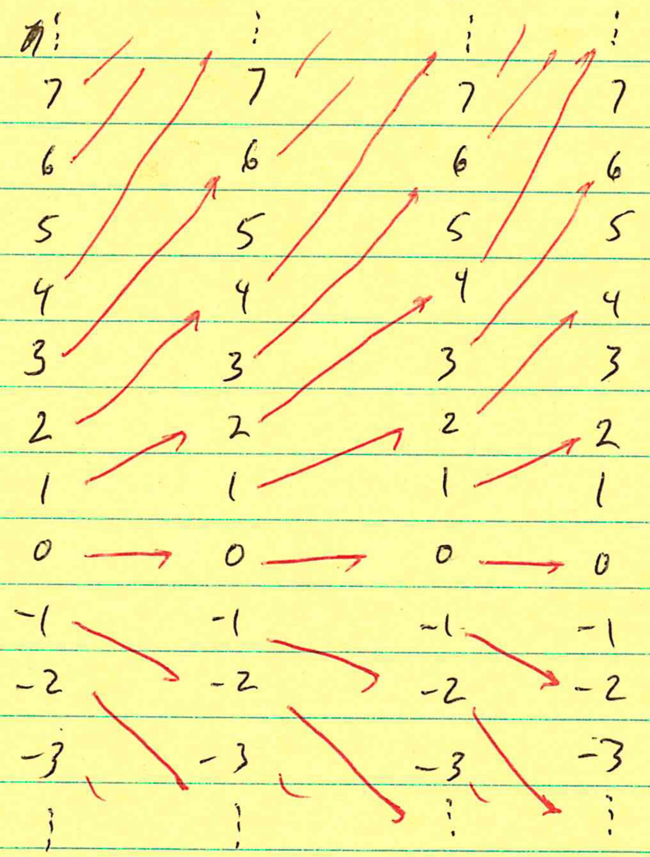
Study the next example carefully.

__Ex__ Let $J = \mathbb{Z}^+$, $G_i = \mathbb{Z}$, $f_{ij} = \times 2^{j-i}$. Then

$$\varinjlim G_i \cong D = \left\{ \frac{k}{2^n} \;\middle|\; k \in \mathbb{Z},\ n \in \mathbb{Z}^+ \cup \{0\} \right\}$$

(The dyadic rationals)

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \cdots$$



__Ex__ Think of $\frac{7}{16}$ as being 7 in $G_5 = \mathbb{Z}$.

Think of $\frac{5}{8}$ as being 5 in $G_4 = \mathbb{Z}$.

Then $\frac{7}{16} + \frac{5}{8} = \frac{7}{16} + \frac{10}{16} = \frac{17}{16}$ is 17 in G_5.

__Ex__ $J = \mathbb{Z}^+$, $G_i = \mathbb{Z}$, $f_{j\,j+1} = \times(j+1)$, $f_{ij} = \times(i+1)(i+2)\cdots(j)$.

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \xrightarrow{5} \mathbb{Z} \xrightarrow{6} \mathbb{Z} \xrightarrow{7} \mathbb{Z} \longrightarrow \cdots\cdots$$

Show that $\varinjlim G_i \cong \mathbb{Q}$.

<u>Def</u> Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a space X. Then $\mathcal{B}$ is a <u>refinement</u> of $\mathcal{A}$ if $\forall B \in \mathcal{B}, \exists A \in \mathcal{A}$ s.t. $B \subset A$. (See Example 2, page 433.)

<u>Def</u> Let $\mathcal{A}$ be a collection of subsets of a space X. We define an abstract simplicial complex (see §3) denoted $N(\mathcal{A})$ called the <u>nerve</u> of $\mathcal{A}$.

The vertices are the individual sets in $\mathcal{A}$. The p-simplicies are all subcollections $\{A_0, \dots, A_p\} \subset \mathcal{A}$ s.t. $\bigcap_{i=0}^{p} A_i \neq \phi$.

The boundary map $\partial_{p+1}(\{A_0, \dots, A_{p+1}\}) = \sum_{i=0}^{p+1} (-1)^i \{A_0, \dots \hat{A}_i, \dots, A_{p+1}\}$.

It is easies to check $\partial\partial = 0$. Thus homology and cohomology group of $N(\mathcal{A})$ are defined.

Now suppose $\mathcal{B}$ is a refinement of $\mathcal{A}$. Let $g: \mathcal{B} \to \mathcal{A}$ be s.t. $g(B) = A$ for some $A \supset B$. Then g induces a simplicial map $g_{\#}: N(\mathcal{B}) \to N(\mathcal{A})$.

If $g': \mathcal{B} \to \mathcal{A}$ is another such map it is <u>contiguous</u> (see pg 67) to g since:

If $g'(B) = A'$ and $g(B) = A$, then $A' \cap A \neq \phi$. Hence $\{A, A'\} \in N(\mathcal{A})$ (it is a 1-simplex). Thus $\exists$ a complex of $N(\mathcal{A})$ whose boundary is $g'(B) - g(B)$.

Now $g: \mathcal{B} \to \mathcal{A}$ induces homomorphisms

$$g_*: H_k(N(\mathcal{B})) \to H_k(N(\mathcal{A}))$$

and $$g^*: H^k(N(\mathcal{B})) \to H^k(N(\mathcal{A}))\,.$$

These are determined solely by $\mathcal{A}$ and $\mathcal{B}$ and not the choice of g since g is contiguous to any other valid $g': \mathcal{B} \to \mathcal{A}$. (Valid meaning $B \subset g'(B)$).

<u>Def</u> Let X be a top. sp. and let J be all the open covers of X directed by refinement ($\mathcal{A} \leq \mathcal{B}$ if $\mathcal{B}$ ~~is~~ a refinement of $\mathcal{A}$.). Then the <u>Čech cohomology</u> groups of X are

$$\check{H}^k(X) = \varinjlim_{\mathcal{A} \in J} H^k(N(\mathcal{A}))\,.$$

<u>Thm 73.2</u> For a simplicial complex K, $\check{H}^n(|K|) \cong H^n(K)$.

<u>Pf</u> See textbook.

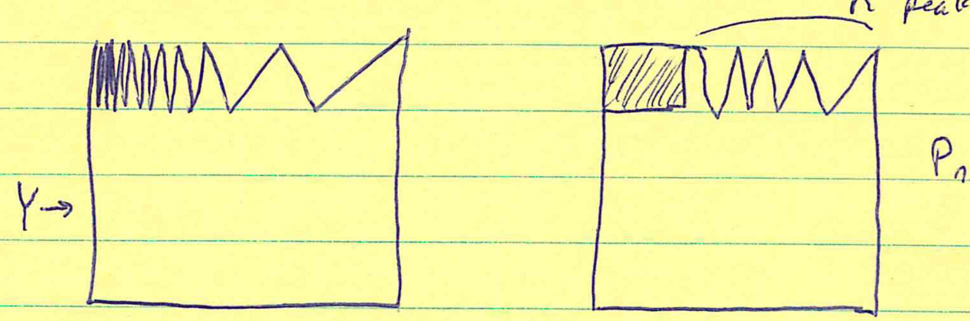
~~Thm 73.4 Let Y be a compact subspace of a normal space X~~.

<u>Thm 73.4</u> Let X be a compact triangulable space. Let $P_1 \supset P_2 \supset P_3 \supset \cdots$ be polyhedra in X and let $Y = \cap P_i$. Then

$$\check{H}^k(Y) \cong \varinjlim_i H^k(P_i)$$

where the maps are induced by inclusion. Same holds for reduced Čech cohomology. Pf: See text book.

Ex Below is a PL version of the topologist's sine curve, and a polyhedron P_n containing it.



Then $P_{n+1} \subset P_n$ and $Y = \bigcap P_n$.

One can show that in simplicial homology and cohomology
$$H_1(Y) = 0 \quad \text{and} \quad H^1(Y) = 0.$$
The textbook does this formally, but it should be "obvious" there are no 1-cycles or 1-cocycles in Y.

But for each n, $H^1(P_n) \cong \mathbb{Z}$ (P_n is homotopic to S^1). The inclusion maps $P_{n+1} \hookrightarrow P_n$ induce identity isomorphisms. Thus

$$\check{H}^1(Y) = \varinjlim H^1(D_n) \cong \mathbb{Z}.$$

Thus, Čech cohomology "sees" this cocycle that ordinary cohomology does not.

<u>Exercise #5</u> Let $X = S^1 \times D^2$, the solid torus. Let $f: X \to X$ be a cont. map that does this:

Let $X_1 = f(X)$, $X_2 = f(X_1)$, etc, $X_{n+1} = f(X_n)$. These are nested solid tori inside X. Let

$$S = \cap X_n$$

S is called the <u>soleniod</u> or sometimes the <u>Smale-Williams soleniod</u>. It is an important example in dynamical systems.

Each X_n is homeo. to a polyhedron. Clearly $H^1(X_n) \cong \mathbb{Z}$ $\forall n$. The maps induced by inclusion are $\times 2$. Hence

$$\check{H}^1(S) \cong D \text{, the dyadic rationals.}$$

See https://en.wikipedia.org/wiki/Solenoid_(mathematics)