Section 7 H_0 and $\tilde{H}_p$.

Def (pg 31) Let c_1 and c_2 be p-chains such that $c_1 - c_2 \in B_p$. Then we say c_1 and c_2 are homologous and write $c_1 \sim c_2$. If $c \sim 0$ we say c is homologous to zero or null homologous.

Def (pg 11) If w is a vertex of K, then the star of w, denoted $st(w)$, is the union of the interiors of every simplex that has w as a vertex.
(The interior of a simplex was defined on pg 5.)
$St\,w$ is open in $|K|$.

Ex

Thm 7.1 Let K be a complex, possibly infinite. For each component of $|K|$ choose a vertex, v_α. Then $H_0(K)$ is a free abelian group whose generators can be represented by $\{v_\alpha\}$,

$$H_0(K) = \langle [v_\alpha] \rangle_{\text{abelian}}$$

If there are n components of $|K|$, then

$$H_0(K) \cong \mathbb{Z}^n.$$

pf For any vertex v of K define $C_v = \bigcup\{\mathrm{St}\, w \mid w \sim v\}$.
We claim the distinct C_v are the ~~path~~ components of $|K|$.
We outline the proof of this.

(i) C_v is open in $|K|$.
(ii) $C_v = C_{v'}$ if $v \sim v'$.
(iii) C_v is path connected (and hence connected)
(iv) If $C_v \neq C_{v'}$ then $C_v \cap C_{v'} = \emptyset$.

Just notice that $v \sim w$ means there is a finite chain of edges whose union is a path from v to w.
See textbook for details.

Now to prove the thm, by the axiom of choice $\exists$ $\{v_\alpha\}$, a collection of vertices with just one vertex from each distinct C_v.

Let w be a vertex of K. Then $w \in C_{v_\alpha}$ for one and only one of the v_α's. Thus $w \sim v_\alpha$. $\exists$ a 1-chain

$$p = [\underset{\underset{v_\alpha}{\shortparallel}}{a_0}, a_1] + [a_1, a_2] + \cdots + [a_{n-1}, \underset{\underset{w}{\shortparallel}}{a_n}]$$

with $\partial p = w - v_\alpha$. Therefore, any 0-chain is homologous to a linear combination of v_α's.
Thus $\{[v_\alpha]\}$ generates $H_0(K)$.

But, could there be relations?

Suppose $c = \sum n_\alpha v_\alpha = \partial d$ for some 1-chain d.
We will show each $n_\alpha = 0$. Let $d_\alpha =$ "$d \cap C_{v_\alpha}$".
Then $d = \sum_\alpha d_\alpha$.

Def Now we make a definition that will be used later.
Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be given by

$$\varepsilon\left(\sum n_\alpha v_\alpha\right) = \sum n_\alpha .$$

It is called the augmention map and is a group homomorphism.

We had $d = \sum_\alpha d_\alpha$. Thus $\partial d = \sum_\alpha \partial d_\alpha$.

Each ∂d_α is carried by C_{v_α}, thus $\partial d_\alpha = n_\alpha v_\alpha$, for each α.

For any 1-chain ε takes its ∂ to 0. (For any 1-simplex $[a,b]$, $\varepsilon(\partial[a,b]) = \varepsilon(b-a) = 1-1 = 0$; then use induction.) Thus for each α

$$\varepsilon(\partial d_\alpha) = 0 \Rightarrow n_\alpha = 0. \quad \blacksquare$$

Reduced Homology Groups.

We just observed that $\varepsilon \circ \partial_1 = 0$, since $\partial_1(e) = v_i - v_j$ for any edge. Thus $\text{im}\, \partial_1 \subset \ker \varepsilon$. We define

$$\tilde{H}_0(K) = \frac{\ker \varepsilon}{\text{im}\, \partial_1}$$

And for convenience we define $\tilde{H}_p(K) = H_p(K)$ for $p \geq 1$. These are called the reduced homology groups of K.

In other words the sequence

$$\to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

gives the usual homology groups, while

$$\to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

gives the reduced homology groups.

See next page.

Thm 7.2 $\tilde{H}_0(K) \oplus \mathbb{Z} \cong H_0(K)$

Pf Case 1 Suppose $|K|$ has finitely many components. Select one vertex from each: $v_1, v_2, \dots, v_n$. Any 0-chain is homologous to one of the form $\sum_{i=1}^{n} m_i v_i$.

The kernel of ε is generated by $\{v_1 - v_2, v_1 - v_3, \dots, v_1 - v_n\}$.
Pf: Let $\sum_{i=1}^{n} m_i v_i \in \ker \varepsilon$. Then $\sum_{i=1}^{n} m_i = 0$. Thus,

$$\sum_{i=1}^{m} m_i v_i = \sum_{i=2}^{n} -m_i (v_1 - v_i).$$

None of $v_1 - v_i$ are null homologous since there is no 1-chain from v_1 to v_i (for $i \neq 1$).
Therefore

$$\tilde{H}_0(K) = \langle v_1 - v_2, v_1 - v_3, \dots, v_1 - v_n \rangle \cong \mathbb{Z}^{n-1}.$$

Case 2 If $|K|$ has infinitely many components the above still works since each 0-chain is a finite sum. If $\{v_\alpha\}$ is a collection of vertices with one in each component of $|K|$, select one and call it v_{α_0}. Then

$$\tilde{H}_0(K) = \langle v_{\alpha_0} - v_\alpha \mid \text{all } \alpha \neq \alpha_0 \rangle.$$

This is not very interesting since $\mathbb{Z}^X \oplus \mathbb{Z} \cong \mathbb{Z}^X$ when X is an infinite cardinal.