

## Tensor Products of Abelian Groups<sup>1</sup>

We will write abelian groups additively and use 0 for the identity element.

Let  $A$  and  $B$  be abelian groups. Recall that their **direct sum** is given by

$$A \oplus B = (A \times B, (a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2)).$$

Recall also that a map  $h : A \oplus B \rightarrow G$  (another abelian group) is a **homomorphism** if

$$h((a_1, b_1) \oplus (a_2, b_2)) = h(a_1, b_1) + h(a_2, b_2).$$

Here is another way to think about direct sums. Let  $F(A, B)$  be the free abelian group generated by the elements of  $A \times B$ . Let  $R_1(A, B)$  be the subgroup generated by elements of the form  $(a_1 + a_2, b_1 + b_2) - (a_1, b_1) - (a_2, b_2)$ . Then

$$A \oplus B \cong F(A, B)/R_1(A, B).$$

**Exercise.** Check this.

Notice that if  $h : A \times B \rightarrow G$  is simultaneously linear in both variables, *i.e.*,

$$h((a_1 + a_2, b_1 + b_2)) = h(a_1, b_1) + h(a_2, b_2),$$

then  $h$  determines a homomorphism from  $A \oplus B$  to  $G$  and vice versa.

**Exercise.** Check this.

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<sup>1</sup>This is a first draft of lecture notes mostly from §50 of Munkres' Elements of Algebraic Topology. Let me know if you find errors. May 22, 2020

Suppose we are instead interested in maps  $h : A \times B \rightarrow G$  that are **bilinear**, that is they are linear in each variable separately:

$$\begin{aligned} h(a_1 + a_2, b) &= h(a_1, b) + h(a_2, b), \\ h(a, b_1 + b_2) &= h(a, b_1) + h(a, b_2). \end{aligned}$$

**Example.** The map  $(a, b) \mapsto a + b$  from  $\mathbb{Z} \times \mathbb{Z}$  to  $\mathbb{Z}$  is a homomorphism and is not bilinear. The map  $(a, b) \mapsto ab$  is bilinear but is not a homomorphism.

Define  $F(A, B)$  as before. Let  $R_2(A, B)$  be the subgroup generated by all elements of the forms

$$\begin{aligned} (a_1 + a_2, b) - (a_1, b) - (a_2, b), \\ (a, b_1 + b_2) - (a, b_1) - (a, b_2). \end{aligned}$$

Then we define the **tensor product** of  $A$  and  $B$  to be

$$A \otimes B = F(A, B) / R_2(A, B).$$

Notice every bilinear function from  $A \times B$  to  $G$  determines a homomorphism from  $A \otimes B$  to  $G$  and vice versa.

**Exercise.** Check this.

Let  $a \otimes b$  denote the equivalence class of  $(a, b)$  in  $A \otimes B$ . Then  $A \otimes B$  is the set of finite sums of elements of the form  $a \otimes b$ , that is to say,  $A \otimes B$  is generated by  $\{a \otimes b \mid a \in A, b \in B\}$ .

Note the contrast with  $A \oplus B$ . It is generated by  $\{(a, 0), (0, b) \mid a \in A, b \in B\}$  and is equal to  $\{(a, b) \mid a \in A, b \in B\}$ .

**Quick Facts.** It is obvious that

$$(a_1 + a_2) \otimes b = (a_1 \otimes b) + (a_2 \otimes b),$$

$$a \otimes (b_1 + b_2) = (a \otimes b_1) + (a \otimes b_2).$$

Using 0 for the identity in  $A \otimes B$  it is immediate that  $0 = 0 \otimes 0$ . But notice that

$$(0 \otimes b) + (a \otimes b) = (0 + a) \otimes b = a \otimes b \implies 0 \otimes b = 0,$$

$$(a \otimes 0) + (a \otimes b) = a \otimes (0 + b) = a \otimes b \implies a \otimes 0 = 0.$$

It follows that  $G$  tensor the trivial group is trivial.

**Claim.**  $(-a) \otimes b = -(a \otimes b) = a \otimes (-b)$

*Proof.*  $(-a) \otimes b + (a \otimes b) = (-a + a) \otimes b = 0 \otimes b = 0$  and  $a \otimes (-b) + (a \otimes b) = a \otimes (-b + b) = a \otimes 0 = 0$ .

**Claim.**  $(na) \otimes b = n(a \otimes b) = a \otimes (nb), \forall n \in \mathbb{Z}$ .

*Proof.* Use induction for  $n \geq 1$ . We have  $n = 0$ . Use the previous claim for  $n \leq -1$ .

**Theorem.**  $\mathbb{Z} \otimes G \cong G$ .

*Proof.* The map  $h : \mathbb{Z} \times G \rightarrow G$  given by  $h(n, g) = ng$  is bilinear. Let  $\phi : \mathbb{Z} \otimes G \rightarrow G$  be determined by  $\phi(n \otimes g) = ng$ . Then  $\phi$  is the homomorphism induced by  $h$ . Check this.

Let  $\psi : G \rightarrow \mathbb{Z} \otimes G$  be given by  $\psi(g) = 1 \otimes g$ . It is easy to check that  $\psi$  is a homomorphism. We show it is the inverse of  $\phi$ .

$$\psi\phi(n \otimes g) = \psi(ng) = 1 \otimes ng = n \otimes g.$$

$$\phi\psi(g) = \phi(1 \otimes g) = 1 \cdot g = g.$$

□

**Example.**  $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}$  is trivial. *Proof.* It is generated by  $1 \otimes 1$  and  $1 \otimes 2$ . But

$$1 \otimes 2 = 2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0$$

and

$$1 \otimes 1 = -(1 \otimes 1) = 1 \otimes (-1) = 1 \otimes 2 = 0.$$

**Example.**  $\mathbb{Q} \otimes (\mathbb{Z}/n\mathbb{Z})$  is trivial for  $n > 0$ . *Proof.* Obvious for  $n = 1$ . In general

$$r \otimes p = \frac{nr}{n} \otimes p = \frac{r}{n} \otimes np = \frac{r}{n} \otimes 0 = 0.$$

Since all the generators are the identity the group is trivial.

**Example.**  $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$ . *Proof.* Notice

$$\frac{p}{q} \otimes \frac{m}{n} = \frac{mnp}{nq} \otimes \frac{1}{n} = \frac{mp}{nq} \otimes 1.$$

Define  $h : \mathbb{Q} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$  by  $h(\frac{p}{q} \otimes \frac{m}{n}) = \frac{mp}{nq}$  and extending to a homomorphism. It is clearly onto. Suppose  $h(x \otimes y) = 0$ . Then  $xy = 0$  so  $x = 0$  or  $y = 0$ . Hence  $x \otimes y = 0$ . To find the kernel set

$$h\left(\sum x_i \otimes y_i\right) = 0.$$

But,

$$\sum x_i \otimes y_i = \sum x_i y_i \otimes 1 = \left(\sum x_i y_i\right) \otimes 1 = 0.$$

Hence the kernel of  $h$  is trivial and so  $h$  is an isomorphism.

**Example.**  $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ . *Proof.* The only generator is  $1 \otimes 1$  and we notice that  $2(1 \otimes 1) = 0 \otimes 1 = 0$ . To prove the result is not trivial let  $h : \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  be determined by  $h(a \otimes b) = ab$ . It is an isomorphism since it is onto and  $ab = 0$  iff  $a$  or  $b$  is zero.

**Exercise.** Simplify  $\mathbb{Z}/6\mathbb{Z} \otimes \mathbb{Z}/9\mathbb{Z}$ . Do not look at the next theorem! Hint: As a first step write out the generators and show they form three equivalence classes one of which is 0.

**Theorem.**  $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$  where  $d = \gcd(m, n)$ .

*Proof.* Let  $h(a \otimes b) = ab \pmod{d}$ . It extends to a homomorphism since  $(a, b) \rightarrow ab \pmod{d}$  is bilinear. It is onto since  $d$  is less than or equal to  $m$  and  $n$ . Let  $d = xm + yn$ . Assume  $ab = 0 \pmod{d}$ . Then  $d|ab$  and  $ab = dp = mxp + ynp$ . Hence,

$$\begin{aligned} (a \otimes b) &= (mxp + ynp) \otimes 1 = (mxp \otimes 1) + (ynp \otimes 1) = \\ &= (0 \otimes 1) + (1 \otimes ynp) = 0 + 0 = 0. \end{aligned}$$

Thus if  $h(\sum(a_i \otimes b_i)) = 0$  we know

$$h\left(\left(\sum a_i b_i\right) \otimes 1\right) = 0 \implies \left(\sum a_i b_i\right) \otimes 1 = 0.$$

□

**Theorem.**

- (a)  $A \otimes B \cong B \otimes A$ .
- (b)  $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ .
- (c)  $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$ .

See textbook Theorem 50.6 for proofs.

**Corollary.** Assume  $A$  and  $B$  are finitely generated free abelian groups. If  $\{a_1, a_2, \dots, a_m\}$  generates  $A$  and  $\{b_1, \dots, b_n\}$  generates  $B$  then  $\{a_i \otimes b_j \mid i = 1, \dots, m, j = 1, \dots, n\}$  generates  $A \otimes B$  which is also free abelian. (You may have seen a similar result for tensor products of vector spaces.)

**Corollary.** Recall that  $\mathbb{Z}^m \oplus \mathbb{Z}^n \cong \mathbb{Z}^{m+n}$ . But for tensor products we have  $\mathbb{Z}^m \otimes \mathbb{Z}^n \cong \mathbb{Z}^{mn}$ .

**Exercise.** Express  $(\mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/12) \otimes (\mathbb{Z}^3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/10)$  in Smith normal form.

*Answer.*  $\mathbb{Z}^6 \oplus (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/6)^3 \oplus \mathbb{Z}/12 \oplus (\mathbb{Z}/60)^2$ .

**Definition.** Suppose  $f : A \rightarrow B$  and  $g : C \rightarrow D$  are homomorphisms of abelian groups. Let

$$f \otimes g : A \otimes C \rightarrow B \otimes D$$

be determined by  $f \otimes g(a \otimes c) = f(a) \otimes g(c)$  and extending to a homomorphism. It is called the **tensor product** of  $f$  and  $g$ .

**Theorem.** The isomorphism  $h : \mathbb{Z} \otimes G \rightarrow G$  determined by  $h(n \otimes g) = ng$  is natural wrt homomorphism of  $G$ .

*Proof.* Let  $f : G \rightarrow H$  be a homomorphism. We claim that the diagram

$$\begin{array}{ccc} \mathbb{Z} \otimes G & \xrightarrow[\cong]{h} & G \\ \text{id} \otimes f \downarrow & & \downarrow f \\ \mathbb{Z} \otimes H & \xrightarrow[\cong]{h'} & H \end{array}$$

commutes. This is easy. We compute.

$$h'(\text{id} \otimes f(n \otimes g)) = h'(n \otimes f(g)) = nf(g).$$

$$f(h(n \otimes g)) = f(ng) = nf(g).$$

□

Next we explore exact sequences.

**Definition.** Given a sequence of abelian groups and homomorphisms

$$\rightarrow A_i \xrightarrow{\phi_i} A_{i-1} \rightarrow$$

and an abelian group  $G$  we define the **tensor product** of the sequence with  $G$  to be

$$\rightarrow A_i \otimes G \xrightarrow{\phi_i \otimes \text{id}} A_{i-1} \otimes G \rightarrow$$

**Theorem.** If the sequence  $A \rightarrow B \rightarrow C \rightarrow 0$  is exact then so is its tensor product with  $G$ .

*Proof.* Exactness at  $C \otimes G$  amounts to showing  $\psi \otimes \text{id}$  is onto. We will show this indirectly by computing  $\ker(\psi \otimes \text{id})$  and defining a certain isomorphism.

We claim  $\ker(\psi \otimes \text{id})$  is generated by  $K = \{b \otimes g \mid b \in \ker \psi \text{ or } g = 0\}$ . Let  $b \otimes g$  be in this set. Then

$$\psi \otimes \text{id}(b \otimes g) = \psi(b) \otimes g$$

which is 0 is either  $\psi(b) = 0$  or  $g = 0$ . Thus  $\langle K \rangle \subset \ker(\psi \otimes \text{id})$ . Hence,  $\exists$  a homomorphism

$$\Psi : (B \otimes G) / \langle K \rangle \rightarrow C \otimes G$$

induced by  $\psi \otimes \text{id}$ . We will show that  $\Psi$  is an isomorphism by constructing an inverse  $\Theta : C \otimes G \rightarrow (B \otimes G) / \langle K \rangle$ . This will show that  $\langle K \rangle = \ker(\psi \otimes \text{id})$  and that  $\psi \otimes \text{id}$  was onto.

Let  $\theta : C \times G \rightarrow (B \otimes G) / \langle K \rangle$  be given by

$$\theta(c, g) = b \otimes g + \langle K \rangle,$$

where  $b \in \psi^{-1}(c)$ . We show that  $\theta$  is well defined. Let  $b' \in \psi^{-1}(c)$ . Then

$$b \otimes g - b' \otimes g = (b - b') \otimes g \in K.$$

Thus,  $\theta$  is well defined. It is easy to check that  $\theta$  is bilinear. Let  $\Theta$  be the homomorphism induced by  $\theta$ .

Now we compute.

$$\Psi\Theta(c \otimes g) = \Psi((b \otimes g) + \langle K \rangle) = \psi(b) \otimes g = c \otimes g.$$

$\Theta\Psi(b \otimes g + \langle K \rangle) = \Theta(\psi(b) \otimes g) = b' \otimes g + \langle K \rangle = b \otimes g + \langle K \rangle$ , since  $\psi(b) = \psi(b')$ . Thus, we have exactness at  $C \otimes G$ .

Exactness at  $B \otimes G$  is now easy. This image of  $\phi \otimes \text{id}$  is a subgroup  $H$  generated by all the elements of the form  $\phi(a) \otimes g$ . But,  $\text{im } \phi = \ker \psi$  so  $H = \langle K \rangle$ .  $\square$

**Theorem.** If the sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is exact and splits then its tensor product with  $G$  is exact and splits.

*Proof.* See textbook, Theorem 50.4. □

It is not true in general that the tensor product of  $G$  with a short exact sequence is exact when the sequence does not split. However, it is true if  $G$  is torsion free. See Corollary 50.7. Compare to Exercise 4 in §41.

The textbook also covers tensor products of modules. Read on your own.