532 Lecture Notes
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Preface

Different subsets of this material were used in short courses I gave at Tokyo Tech in January & February 2013 and in July & August 2013 at Southern Illinois University Carbondale.

Students had a wide variety of backgrounds so there is a review of topological manifolds, homotopy and homology. Then there is a detailed proof of a classic result of Louis Moser. It describes which 3-manifolds arise from Dehn surgery on a torus knot.

The following notations are used.

$\sim$ means homologous 1-chains or homotopic curves
$\simeq$ means homeomorphic spaces
$\cong$ means fiber-wise homeomorphic fibered spaces
$\equiv$ means isomorphic groups

The rest of the course, not included here, applied Moser's theorem to the study of nonsingular Smale flows covering work by Bin Yu, Elizabeth Haynes and myself. Then we went on to cover some other topics involving flows on 3-manifolds.

- Visually building Smale flows in S^3. Topology Appl. 106 (2000), no. 1, 119.
- Bin Yu, Lorenz like Smale flows on three-manifolds. Topology Appl. 156 (2009), no. 15, 24622469.
- Hayne & Sullivan, preprint.
- Krystyna Kuperberg, Counterexamples to the Seifert conjecture. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 831840.

Contents

Chapter 1

A Quick Introduction to Topology

1 Continuity

Topology is the abstract study of continuity. In calculus courses a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. In analysis students learn that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = c$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in (c - \delta, c + \delta) \Rightarrow f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$.

Exercise 1.1. Prove that these two definitions are equivalent. You may need to review the definition of a limit.

Exercise 1.2. What would "go wrong" if we replaced "...$\exists \delta > 0$..." with "...$\exists \delta \geq 0$..." in the definition of continuity? Which functions would be continuous?

We say f is continuous on a subset $D \subset \mathbb{R}$ if it is continuous at each point in D.

Let X and Y be sets and let $f : X \to Y$ be a function. The question we ask is, what are the simplest structures we need to impose on X and Y for the statement "$f : X \to Y$ is continuous" to be meaningful? Notice we could modify the definition of continuity as follows. Let $f : \mathbb{R} \to \mathbb{R}$. Then we say f is continuous at $x = c$ if for all open intervals I containing $f(c)$ there is an open interval J containing c such that $f(J) \subset I$.

Exercise 1.3. Show that this really is equivalent to the previous definition of continuity. These open intervals need not be symmetric about c or $f(c)$. Does that really matter? What would go wrong if we used closed sets, including single point sets, instead of open intervals? In fact we don't really need open intervals. We could let I and J be more general open sets. Right? See the definition of open sets below.

By definition a set $U \subset \mathbb{R}$ is open if and only if it is the union of open intervals. The empty set ϕ is regarded as open since it is a vacuous union. A subset $C \subset \mathbb{R}$ is closed if $\mathbb{R} - C$ is open. Thus $\mathbb{R}$ and ϕ are both open and closed.

If we have a notion of which subsets of X and Y are "open" then we can generalize the definition of continuity. Let $\mathcal{O}$ be a collection of subsets of X. What properties should it have to earn the title of "the open subsets of X"? On $\mathbb{R}$ the open sets where defined as unions of open intervals. Thus any union of open sets is an open set. We shall require $\mathcal{O}$ to have this property. Another property of open sets in the real line is that finite intersections of open sets are open. We leave the proof to you. But notice infinite intersections of open sets need not be open:

$$\bigcap_{i=1}^{\infty}(-1/n, 1/n) = \{0\}$$

which is not open. But the finite intersection property is important and we shall require $\mathcal{O}$ have it. It is also useful to require that ϕ and X be in $\mathcal{O}$. This turns out to be all we need. These lead us to the following definitions.

Definition 1.1. A set X together with a collection of subsets $\mathcal{O}$ is a *topological space* if the following hold.

1. X and ϕ are in $\mathcal{O}$.

2. The union of any subcollection of $\mathcal{O}$ is in $\mathcal{O}$.

3. The intersection of any finite subcollection of $\mathcal{O}$ is in $\mathcal{O}$.

Definition 1.2. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be topological spaces. Let $f : X \to Y$. Then f is continuous at $c \in X$ if for every open subset V set of Y containing $f(c)$ there is an open subset U of X containing c with $f(U) \subset V$. If f is continuous at for each $c \in D \subset X$ we say f is continuous on D.

Here is an alternative characterization of continuity. Notice that it is a global definition.

Lemma 1.0.1. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous on X if and only if the inverse image of every open subset of Y is an open subset of X.*

Proof. Suppose f^{-1} takes open subsets to open subsets. Let $c \in X$ and let V be any open subset of Y containing $f(c)$. Let $U = f^{-1}(V)$. Then U is an open subset containing c that is mapped into V. Thus f is continuous at c and since c was arbitrary f is continuous on X.

Now suppose f is continuous for every $c \in X$. Let V be an open subset of Y and let $U = f^{-1}(V)$. We need to show U is open. Let $x \in U$. Then V is an open set containing $f(x)$. Thus there is a open subset $U_x \subset X$ containing x such that $f(U_x) \subset (V)$. We can see that $U_x \subset U$ since U is the inverse image of V. Now U is the union of all the U_x for $x \in U$. Hence U is open. □

Example 1.1. Let $f(x) = x^2$ be a function from $\mathbb{R}$ to $\mathbb{R}$. Convince yourself f^{-1} takes open sets to open sets. But notice $f((-1, 1)) = [0, 1)$ is not open.

Example 1.2. We put four different topologies on the real line $\mathbb{R}$ and look at which functions are continuous. Let $\mathcal{U}$ be the usual open sets of $\mathbb{R}$; it is called the **usual topology**. Let $\mathcal{T} = \{\phi, \mathbb{R}\}$; it is called the **trivial topology**. Let $\mathcal{D}$ = all subsets of $\mathbb{R}$; it is called the **discrete topology**. Finally let $\mathcal{F} = \{U \subset \mathbb{R} \,|\, \mathbb{R} - U \text{ is finite}\}|cup\{\phi\}$; it is called the **finite complement topology**. The reader should check that each is a valid topology. Now consider the following.

- Any function from $(\mathbb{R}, \mathcal{D})$ to any topological space is continuous.
- Any constant function from $(\mathbb{R}, \mathcal{U})$ to $(\mathbb{R}, \mathcal{D})$ is continuous but $f(x) = x$ is not.
- Any function from any topological space to $(\mathbb{R}, \mathcal{T})$ is continuous.
- Suppose $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is continuous. Then $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{F})$ is continuous too, but $f : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{U})$ need not be.

2 Interiors and Closures

Let X be a topological space. A subset $C \subset X$ is **closed** if its complement $X - C$ is open.

Exercise 1.4. Show that intersection of closed sets are closed, that finite unions of closed sets are closed, but that infinite unions of closed sets need not be closed.

Let $A \subset X$. The **interior** of A is the largest open set within A and the **closure** of A is smallest closed set that contains A. In symbols

$$\text{int}(A) = \bigcup_{U \in \mathcal{U}} U,$$

where $\mathcal{U}$ is all the subsets of A that are open in X, and

$$\text{cl}(A) = \bigcap_{C \in \mathcal{C}} C,$$

where $\mathcal{C}$ is the set of all closed sets in X that contain A.

Some books use the notations $\overline{A} = \text{cl}(A)$ and $A^o = \text{int}(A)$.

Example 1.3. $\text{int}\left(\text{cl}\left((3,4) \cup (4,7) \right)\right) = (3,7)$.

Exercise 1.5. What is $\text{cl}\left(\text{int}\left([0,4) \cup \{7\}\right)\right)$?

Exercise 1.6. Look up the “Cantor middle thirds set”. What is its interior?

3 Connectedness

Definition 1.3. A subset C of a topological space X is **disconnected** if there exists a pair of open sets U and V such that $U \cap C \neq \phi$, $V \cap C \neq \phi$, $U \cap V = \phi$ and $C \subset U \cup V$. If no such pair exists then C is **connected**.

Theorem 1.1. *It can be shown that the intervals of the real line are connected. [14].*

Exercise 1.7. Prove that the continuous image of a connected space is connected.

Definition 1.4. Let $C \subset Q$. Then C is a **connected component** or just **component** of Q if the only connected subset of Q that contains C is C. The number of components can be finite or infinite.

Definition 1.5. A topological space X is **path connected** if for any two points x and y in X there is a continuous function f (a **path**) from $[0, 1]$ into X such that $f(0) = x$ and $f(1) = y$.

Exercise 1.8. Prove that a path connected space is connected. (The converse is false. Google: "topologist's sine curve.")

Exercise 1.9. The space $\{1, 2, 3\}$ with the trivial topology is connected and even path connected. Prove this. (The trivial topology means the only open sets are ϕ and the entire set.)

Exercise 1.10. Consider $\mathbb{R}$ in the four topologies introduced earlier. Discuss which subsets of $\mathbb{R}$ are connected in each topology.

4 Compactness

Closed bounded subsets of the real line have some important properties. Any continuous function $f : [a, b] \to \mathbb{R}$ has a maximum. That is there must be a number $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. There are several ways this can be generalized to other topological spaces. We will present the most common one which involves the study of open covers of a topological space. Here is an example.

Let $X = [0, 1)$. Then the collection $\mathcal{C} = \{(-1, 1 - \frac{1}{n}) \mid n \geq 1\}$ is an **open covering** of X. Notice that no finite subcollection could cover X. Now consider $\mathbb{R}$ with open covering $\{(-n, n) \mid n \geq 1\}$. Again there is no finite subcover. But for $I = [0, 1]$, or any closed bounded subset of $\mathbb{R}$, any open covering has a finite subcover. We won't prove this here, but try some examples on I. The converse also holds: if a subset of $\mathbb{R}$ is not both closed and bounded then there exists an open covering that does not have a finite subcover. This leads to the following definition.

Definition 1.6. A topological space X is **compact** if every open cover has a finite subcover.

Theorem 1.2. *Let $f : X \to Y$ be continuous. If X is compact so is its image in Y.*

Outline of proof. Take an open cover $\mathcal{V}$ of $f(X)$. Pull it back with f^{-1} to get an open cover of X. It has a finite subcover. Then use the corresponding subcover of $\mathcal{V}$ to get a finite open covering of $f(X)$. □

Exercise 1.11. Give an example of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ that takes a closed set to one that is not closed.

Exercise 1.12. Give an example of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ that takes a bounded set to one that is not bounded.

5 Subspaces and Products

Let $(X, \mathcal{O})$ be a topological space. Let Q be some subset of X. We define a topology on Q as follows. Let $\mathcal{Q} = \{U \cap Q \,|\, U \in \mathcal{O}\}$. In words, a subset of Q is open if it can formed as the intersection of an open subset of X with Q. It is easy to prove that this does give a topology on Q. It is called the **subspace topology**.

Example 1.4. Consider $I = [0, 1] \subset \mathbb{R}$. Then in the subspace topology the set $[0, 0.2)$ is open since $(-1, 0.2) \cap [0, 1] = [0, 0.2)$. Now we can talk about a function on I being continuous or not at the end points. In calculus one typically use limits from the left and right to define continuity at end points of closed intervals. For example $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Exercise 1.13. Let $B \subset A \subset X$, where X is a topological space. Then there are two ways to construct subspace topologies on B; one by using the subspace topology of A first and the other by considering $B \subset X$ directly. Show that these are the same.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces. Suppose we wish to study continuous functions on $X \times Y$. We need a topology for $X \times Y$. At first we might try $\mathcal{B} = \{U \times V \,|\, U \in \mathcal{X} \,\&\, V \in \mathcal{Y}\}$. But this does not quite work. For example consider $\mathbb{R} \times \mathbb{R}$ and the open unit disk $\{(x, y) \,|\, x^2 + y^2 < 1\}$. Surely we want the open disk to be open. Yet it cannot be written in the form $U \times V$ for open subsets of $\mathbb{R}$. Also $\mathcal{B}$ is not closed under even finite unions:

$$(0, 2)^2 \cup (1, 3)^2 \notin \mathcal{B}.$$

(Although $\mathcal{B}$ is closed under finite intersections.) The right definition is this: let $\mathcal{Z}$ be all the possible unions of members of $\mathcal{B}$. Then one can prove that

$\mathcal{Z}$ does give a topological structure for $X \times Y$. It is called the **product topology**.

Exercise 1.14. Let $Q = I \times I$ be the unit square in $\mathbb{R}^2$. We can define a topology for Q is two ways. First, use the product topology of the subspace topologies each of I. Second, use the subspace topology on Q as a subspace of $\mathbb{R} \times \mathbb{R}$ with the product topology. Convince yourself these are the same.

It is known that the product of compact spaces is compact and the product of (path) connected spaces is (path) connected.

6 Homeomorphisms

Now we use the idea of continuity to talk about when two topological spaces are "essentially the same". This is similar to the isomorphism problem in algebra. Let X and Y be topological spaces. Suppose $h : X \to Y$ has the following properties: it is one-to-one, onto, continuous and h^{-1} is continuous. Then h is called a **homeomorphism** and we say X and Y are **homeomorphic** or **topologically equivalent**. If X is homeomorphic to Y we may write $X \simeq Y$.

The major problem in topology is given two topological spaces how can we determine whether or not they are topologically equivalent. First we consider finite sets. If X has m elements and Y has n elements then there cannot be a bijection between them unless $m = n$. This is just counting.

Let $X = \{1, 2, 3\}$. We will put three different topologies on X.

- $\mathcal{T}_1 = \{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$.
- $\mathcal{T}_2 = \{\phi, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$.
- $\mathcal{T}_3 = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$.

Check that these are topologies. If we define $h : X \to X$ by $f(1) = 3$, $f(2) = 2$ and $f(3) = 1$ then you can check that h is a homeomorphism from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$. But there is no such homeomorphism from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_3)$. Here is a proof. Suppose $k : X \to X$ was such a homeomorphism. Then $k^{-1}(\{1\})$ must be a one element member of $\mathcal{T}_1$. Therefore $k^{-1}(1) = 1$. But by the same reasoning we must have $k^{-1}(2) = 1$. Thus k cannot exist.

Notice $\mathcal{T}_1$ has 4 members while $\mathcal{T}_3$ has 5. In general, if two topologies on a finite set are homeomorphic then the number of open sets must be the same in each.

Exercise 1.15. Let $X_n = \{1, 2, 3, \ldots, n\}$. How many topological structures can X_n have? How many are topologically distinct? That is how many topological equivalence classes are there for X_n? This is probably hard. Try working it out for $n = 3$, 4 and 5.

Exercise 1.16. Let X and Y be topological spaces. Suppose m is the number of components of X and n is the number of components of Y. If $X \simeq Y$ show that $m = n$.

Exercise 1.17. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be topological spaces. Let $h : X \to Y$ be a homeomorphism. Show that it induces a bijection between $\mathcal{U}$ and $\mathcal{V}$.

Exercise 1.18. Show that topological equivalence really is an equivalence relation on the collection of all topological spaces. You will need to show the composition of continuous functions in continuous.

7 Cut points

Is $[0, 1)$ homeomorphic to $(0, 1)$? Suppose $h : [0, 1) \to (0, 1)$ is a homeomorphism. If we restrict the domain of h to $(0, 1)$ and call this k then $k : (0, 1) \to (0, 1) - \{h(0)\}$ is a homeomorphism. Check this. But $(0, 1)$ has just one component and its image has two. Contradiction.

Definition 1.7. Let X be a topological space. Let $p \in X$ and let C be the component that contains p; of course it could be that $C = X$. If $C - p$ in the subspace topology is not connected we say that p is a **cut point** of C (or X).

In our example the point $0 \in [0, 1)$ is not a cut point but since every point of $(0, 1)$ is a cut point we derived a contraction. We will do another example.

Example 1.5. We define three sets in $\mathbb{R}^2$ and give each the subspace topology. Let $A = [-1, 1] \times \{0\}$, $B = \{(x, y) \,|\, x^2 + y^2 = 1\}$ and $C = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$. Show no two of these are homeomorphic.

Solution. Suppose $h : A \to B$ is a homeomorphism. Let $A' = A - \{(0,0)\}$, $B' = B - \{h(0,0)\}$ and let h' be the restriction of h to A'. But this is impossible since A' has two components and B' has only one no matter where $h(0,0)$ is. A similar argument shows A is not homeomorphic to C. (How would you prove that B or C with one point deleted is still connected?)

Now suppose $g : B \to C$ is a homeomorphism. Let $B' = B - \{(1,0),(-1,0)\}$, let $C' = C - \{g(1,0), g(-1,0)\}$ and let g' be the restriction of g to B'. But now g' would be a homeomorphism from a space with two components to a connected space. □

There are limits to this method. We can distinguish between $\mathbb{R}$ and $\mathbb{R}^n$ for any $n > 1$ but not between $\mathbb{R}^2$ and $\mathbb{R}^3$ since both remain connected when a finite number of points are deleted.

Chapter 2

Manifolds

1 Definitions

An **n-dimensional manifold without boundary**, M, is a topological space such that for each point $x \in M$ there exists an open set containing x that is homeomorphic to an open ball in $\mathbb{R}^n$. (We may assume the homeomorphism is to the open unit ball centered at the origin and takes x to the origin. [10]) If there are points y in M for which this fails but for which there is a subset H of M containing y and a homeomorphism

$$h : H \to \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 < 1 \,\text{and}\, x_1 \geq 0\}$$

taking y to the origin, then M is a **n-dimensional manifold with boundary**. Such points y form the **boundary** of M which is denoted ∂M. The **interior** of M is $\text{int}(M) = M - \partial M$.[1]

Here are some standard examples of manifolds. The unit interval $I = [0, 1]$ is a 1-manifold with $\partial I = \{0, 1\}$. A **circle** or 1-**sphere**, also denoted S^1, is any space homeomorphic to $\{(x, y) \mid x^2 + y^2 = 1\}$. A **2-disk**, D^2, is a any space homeomorphic to the closed unit disk $\{(x, y) \mid x^2 + y^2 \geq 1$. Notice that $\partial D^2 = S^1$ and $\partial S^1 = \phi$. A **2-sphere**, S^2, is a any space homeomorphic to the unit sphere in $\mathbb{R}^3$, $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. A **3-ball**, B^3, is a any space homeomorphic to the closed unit ball in $\mathbb{R}^3$, $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$. A **3-sphere**, S^3, is any space homeomorphic to $\{(w, x, y, z) \mid w^2 + x^2 + y^2 + z^2 = 1\}$ as a subspace of $\mathbb{R}^4$. The torus T^2 is $S^1 \times S^1$ or any space homeomorphic to

[1] Manifolds are also assumed to be Hausdorff and second countable.

this. The n-spheres and the torus do not have boundary. All of these spaces are path connected.

2 Gluing, Connected Sums and Compactification

We won't be precise in our definitions here but will proceed by examples. If we "identify" the end points of the unit interval we get a new manifold that is homeomorphic to the circle. If we take the square $I \times I$ and identify each point on the bottom edge with the point on the top edge that is above it we get a new manifold that is homeomorphic to a cylinder. We say that we have glued the top and bottom edges. If instead of gluing $(x, 0)$ to $(x, 1)$ we glued $(x, 0)$ to $(1-x, 1)$ the result would be a Möbius band! If we glue $(x, 0)$ to $(x, 1)$ and $(0, y)$ to $(1, y)$, for $x, y \in I$, the result is a torus.

Exercise 2.1. Explain why a Möbius band is not homeomorphic to an annulus but a strip with a full (360^o) twist is.

If we take two closed disks and identify their boundaries the result is a 2-sphere. If we take two closed 3-dimensional balls, B_1 and B_2, and identify points on their boundary 2-spheres the resulting 3-manifold without boundary is a 3-sphere. See Figure 2.1. The identification is achieved by choosing a homeomorphism $h : \partial B_1 \to \partial B_2$ and identifying x with $h(x)$ for each $x \in \partial B_1$. It can be proven that the topological type of the result is independent of the choice of h [10].

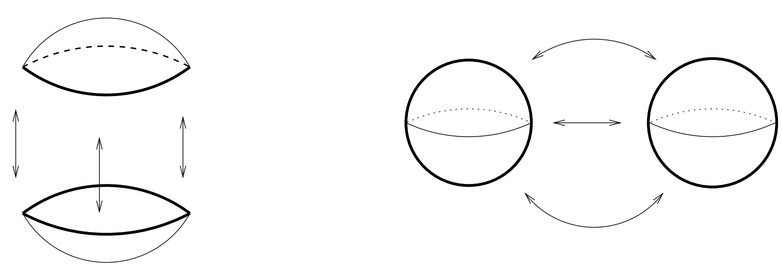

Figure 2.1: Gluing two disks gives a 2-sphere; gluing two balls gives a 3-sphere.

For any two path connected 3-manifolds M_i, $i = 1, 2$, we can form the **connected sum** as follows. Select a closed 3-ball in each that does not meet the boundary (if there is one) and remove their interiors. Now choose a homeomorphism from the new boundary 2-sphere of $M_1 - \text{int } B_1$ to the new boundary 2-sphere of $M_2 - \text{int } B_2$. Glue the two 2-spheres using this homeomorphism. The new manifold is denoted $M_1 \# M_2$ and its topological type is independent of choice of the 3-ball and the homeomorphism [6]. Figure 2.2 illustrates the result of forming the connected sum of two solid tori; it looks like a solid torus with a smaller solid torus carved out of it; the dashed circle represents the 2-sphere where the gluing occurred.

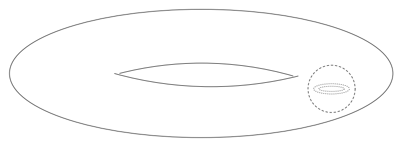

Figure 2.2: The connected sum of two solid tori

If the only way a manifold M can be written as a connected sum is $M \simeq M \# S^3$ then we say M is **prime**. Every compact path connected 3-manifold without boundary can be written uniquely as a connected sum of prime 3-manifolds! [6]

We give another way to construct the 3-sphere this time by gluing two solid tori together. Figure 2.3 shows how to see this starting from gluing two 3-balls together. You decompose one of the 3-balls into a solid torus and a solid cylinder (in the donut hole). We do the gluing in two steps. First glue the top and bottom disks on the cylinder to the other 3-ball. This forms a solid torus. Now glue the two solid tori together and *voila*, we have realized S^3 as the union of two solid tori.

There is yet another way to construct spheres that will be useful for us. Consider the union of the real line $\mathbb{R}$ with a new point called ∞. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Topologize $\overline{\mathbb{R}}$ as follows. Let the open sets be all the open subsets of $\mathbb{R}$ together with sets of the form $\{\infty\} \cup O$ where $\mathbb{R} - O$ is compact. With this topology $\overline{\mathbb{R}}$ is homeomorphic to S^1. This is called the **one point compactification**. The same process can be applied to make $\mathbb{R}^2 \cup \{\infty\}$ homeomorphic to S^2 and $\mathbb{R}^3 \cup \{\infty\}$ homeomorphic to S^3.

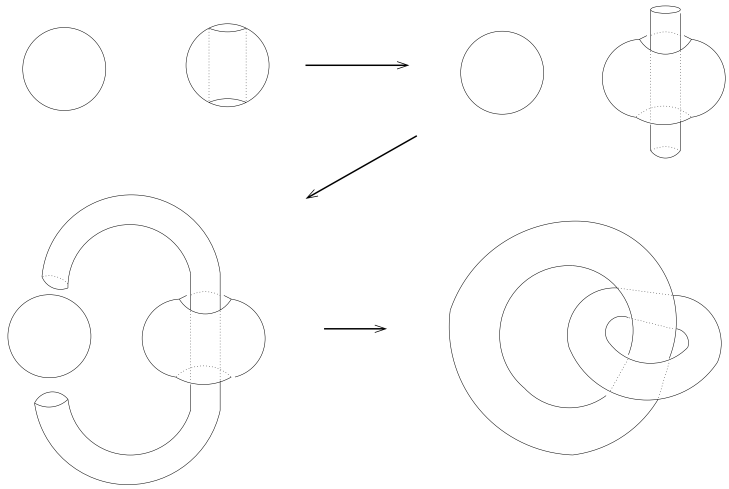

Figure 2.3: Realizing S^3 as the union of two solid tori

3 Knots

A **knot** is a circle embedded in the interior of a 3-manifold, that is there is a homeomorphism $h : S^1 \to K \subset \text{Int } M$. A knot is said to be an **unknot** if it forms the boundary of a disk in M. Thus the unit circle U in the xy-plane in $\mathbb{R}^3$ is unknotted. Two knots K_1 and K_2 in M are regarded as equivalent or as having the same **knot type** if one can be deformed into the other without cutting the knots or the surrounding space. This is formalized by saying they are **ambiently isotopic**, which we define next.

Definition 2.1. Two knots K_1 and K_2 in M are **ambiently isotopic** if there is a continuous function $S : M \times I \to M$ such that $S(x, 0)$ is the identity (hence $S(K_1, 0) = K_1$), $S(K_1, 1) = K_2$ and for each $t \in I$ $S(x, t) : M \to M$ is a homeomorphism.

Sometimes it is useful to give a knot an **orientation**. For us this will just mean picking a preferred direction and indicating it with an arrowhead.

Given a knot K in a 3-manifold M a **tubular neighborhood** of K is a solid torus that misses ∂M and has K as its **core**. It is denoted by $N(K)$. See [18] regarding the existence of tubular neighborhoods. A solid torus whose core is unknotted is said to be **standardly embedded**; likewise for the boundary of such a solid torus.

Let V be a standardly embedded solid torus in S^3 and let $T = \partial V$. A knot in T is called a **torus knot**. The simplest torus knot, besides the unknot, is the **trefoil**; see Figure 2.4.

Although only unknots bound disks every knot in S^3 is the boundary of some orientable (*i.e.*, two sided) surface. Such a surface is called a **Seifert surface**. We won't prove this fact here (see [3]) but Figure 2.4(upper left) shows a Seifert surface for the trefoil.

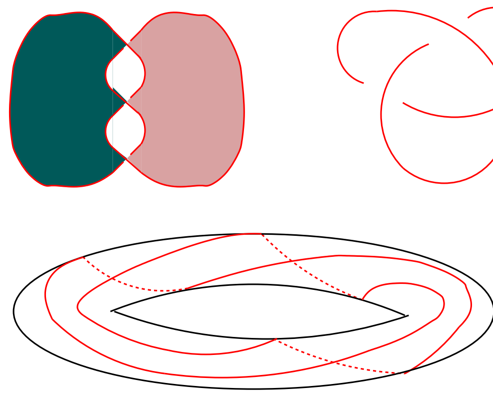

Figure 2.4: Three views of the trefoil knot

Exercise 2.2. Convince yourself that the three curves in Figure 2.4 are ambiently isotopic. Convince yourself that the Seifert surface shown is homeomorphic to a torus with the interior of a closed disk removed.

Let V be a standardly embedded torus in S^3 with $T = \partial V$. A curve on T that bounds a disk in T is considered *trivial*. A nontrivial curve on T that bounds a disk within V is called a **meridian**. A nontrivial curve on T that meets a meridian exactly once transversely[2] is called a **longitude**. A longitude that bounds a disk in the closure of $S^3 - V$ is called a **standard longitude**. A standard longitude and a meridian that meet at only one point form a **standard longitude-meridian pair**; they can be used to put coordinates on T.

Any two meridians of V are ambient isotopic within ∂V. This is not true for longitudes. In Figure 2.5 we show two longitudes where one wraps around

[2] *Transversely* means that one curve passes through the other; the curves cannot be tangent.

the solid torus several times. But any two longitudes are ambient isotopic within V. This subtle point is important to keep in mind. Given any two longitudes on a solid torus V we can find a homeomorphism from V to V that takes one longitude to the other. This involves *twisting* the torus.

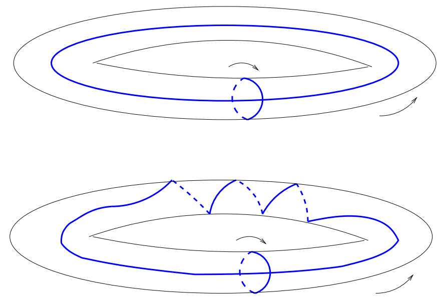

Figure 2.5: Meridian-longitude pairs

If a solid torus is given as being inside S^3 but its core is knotted, we can define a **preferred longitude**. A preferred longitude of a solid torus V in S^3 is a longitude that is the boundary of a Seifert surface of the core of V minus the interior of V. It can be shown that up to ambient isotopy in ∂V there is only one choice for the preferred longitude [3, 18].

If the torus is standardly embedded in S^3 then a preferred longitude will bound a disk in $S^3 - \text{Int } V$ and is easy to visualize; it is the same as a standard longitude. Determining a preferred longitude of a knotted solid torus is not visually obvious. Figure 2.6 shows a preferred longitude for a trefoil solid torus. We will use this later. (A preferred longitude has *linking number* zero with the core [18].)

We will say a little more about torus knots. Let L and M be a standard longitude-meridian pair for standardly embedded solid torus V in S^3. Let $T = \partial V$. Let $h : [0,1)^2 \to T$ be such that the first factor is mapped onto L and the second onto M. A linear function, $y = sx$ from $\mathbb{R}$ to $\mathbb{R}$ can be made into one whose graph is in $[0,1)^2$ by using modulo one arithmetic. It is easy to prove that the graph comes back to (0,0) if and only if the slope s is rational. This image can then to mapped onto T as a torus knot. Let $s = m/l$ and assume it is in reduced form. Then the knot formed in T is called an (l, m)-torus knot. It wraps around the longitude l times and the meridian m times.

Thus L is a $(1,0)$-torus knot and M is a $(0,1)$-torus knot (M has infinite

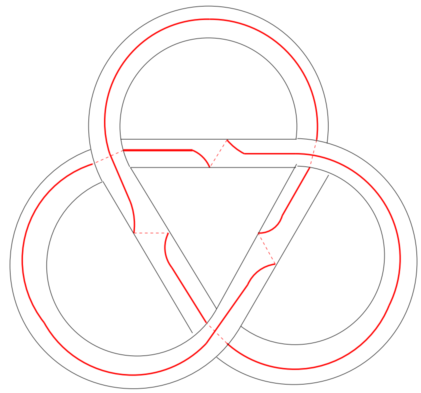

Figure 2.6: A preferred longitude of the trefoil knot based on a figure from [18].

slope). The trefoil in the lower part of Figure 2.4 is a $(2, 3)$-torus knot. It can sometimes be useful to allow $(0, 0)$ to represent a trivial torus knot. If $l = 0$ than m can only be ± 1 and if $m = 0$ then $l = \pm 1$. If neither l or m is zero then the only restriction is that l and m be coprime, that is they have no prime common factors.

If we consider ambient isotopy only within T then there is a one-to-one correspondence between the allowed pairs and the knot types on T up to sign. But thinking of the knots as being in S^3 there are the following equivalences: $(0, 0)$, $(m, \pm 1)$, $(\pm 1, l)$, are unknots, (m, l) is equivalent to (l, m) and changing both signs of l and m does not change the knot type. Changing only one sign gives a mirror image. (A knot is not usually ambiently isotopic to its mirror image, but in some applications the distinction is not important and the definition of knot equivalence is broadened to make mirror images equivalent.)

Exercise 2.3. Which of the knots in Figure 2.7 is equivalent to a trefoil?

Exercise 2.4. For $s = 2/5$, $4/5$ and $-7/3$ plot $y = sx \bmod 1$ on I^2. Then carefully draw these on a torus.

Exercise 2.5. Which torus knots are equivalent if we only allow ambient isotopies within V?

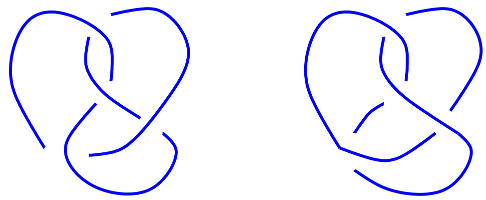

Figure 2.7: Figure for Exercise 2.3

4 Surfaces

The set, up to homeomorphism, of compact, connected, one dimensional manifolds without boundary is $\{S^1\}$. The set, up to homeomorphism, of compact, connected, one dimensional manifolds with nonempty boundary is $\{I\}$. See [10] for the proofs.

For 2-manifolds the problem is a little harder. A 2-manifold is often called a **surface**. The theory behind the topological classification of surfaces is given in detail in many places: see [7, 9] for elementary treatments or [12, 14, 10] for more advanced ones. Here we will just state the main results without proof.

We can break the list of 2-manifolds into two subsets, orientable and nonorientable. We know that S^2 and T^2 are 2-manifolds. They are two-sided and hence orientable. We can make additional 2-manifolds in the following way. Let S be a surface without boundary. Let D_1 and D_2 be disjoint closed disks in S. Remove their interiors. Then attach $S^1 \times I$ to $S - \text{int}(D_1 \cup D_2)$ by gluing $S^1 \times 0$ to ∂D_1 and $S^1 \times 1$ to ∂D_2. This creates a new surface. The process is called **adding a handle**. Let F_n be the surface created by adding n handles to S^2. Then the complete list of compact, connected, orientable surfaces without boundary is $\{S^2 = F_0, T^2 = F_1, F_2, F_3, \ldots\}$.

There is a similar process for creating nonorientable surfaces calling **adding a cross-cap**. Let S be a surface without boundary and remove the interior of one closed disk. Attach a Möbius band to this surface by gluing the boundary of the Möbius band to the boundary where the disk was removed. This creates a new surface. Let G_n be the result of adding n cross-caps to S^2. Then the complete list of compact nonorientable surfaces without boundary is $\{G_1, G_2, \ldots\}$. The first two in this list have names: G_1 is the **projective plane**, denoted P_2 and G_2 is the **Klein bottle** - drink from it at your own risk! (P^2 is sometimes called the **real project plane** and denoted $\mathbb{R}P^2$.)

You might think that adding a cross-cap to F_n would create a nonorientable surface not on the list. But it can be show that adding a cross-cap to F_n gives G_{2n+1}. It is also known that adding a handle to G_n gives G_{n+2}.

To construct compact surfaces with boundary from a surface S we can remove the interiors of a finite number of disjoint disks. In fact this gives all possible compact surfaces with boundary. Thus for any surface with boundary the boundary is a disjoint union of circles.

The **genus** of F_n or G_n is n and the genus of a compact surface with boundary is just the genus of the surface formed by attaching a disk to each boundary component. The **Euler characteristic** of a surface S, denoted $\chi(S)$, is defined in the following way. Build the surface S using polygons. Then

$$\chi(S) = V - E + F$$

where V is the number of vertices, E is the number of edges and F is the number of faces. It can be shown that $\chi(S)$ is independent of the polygon model used for S.

Exercise 2.6. Show that $\chi(F_n) = 2 - 2n$ and that if we remove m disks from F_n then $\chi = 2 - 2n - m$. Develop similar formulas for nonorientable surfaces.

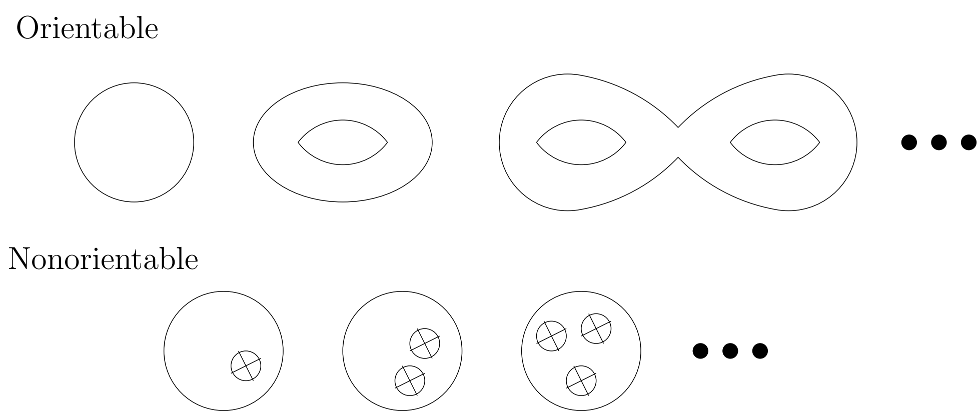

Figure 2.8: Surfaces

The connected sum can be defined for surfaces. Then adding a handle is equivalent to taking the connected sum with a torus. Adding a cross-cap is equivalent to taking the connected sum with a projective plane. Then the list

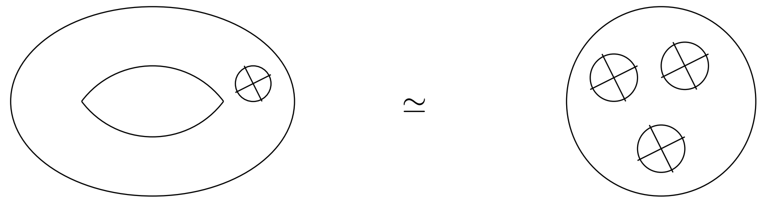

Figure 2.9: Torus with cross-cap $\simeq$ sphere with 3 cross-caps.

of compact connected surfaces without boundary is S^2, $\#_{i=1}^n T_i^2$, and $\#_{i=1}^n P_i^2$ for $n = 1, 2, 3, \ldots$.

5 Lens Spaces

We saw earlier that we can construct S^3 by gluing two solid tori together. If you observe that construction carefully you will see that the gluing map takes a standard meridian-longitude pair on one solid torus to another one on the other but that the meridian maps to a longitude and the longitude maps to a meridian.

What happens if we do the gluing differently? Let $h : \partial V_1 \to \partial V_2$ be a homeomorphism from the boundary of solid torus V_1 to the boundary of solid torus V_2 that takes a meridian to a meridian and a longitude to a longitude. For example h could just be the identity map. What do we get?

Let (L_1, M_1) and (L_2, M_2) be meridian-longitude pairs of V_1 and V_2 respectively. We use the identity map for h, so $h(M_1) = M_2$ and $h(L_1) = L_2$. For $i = 1, 2$ let D_i be a disk in V_i with boundary M_i. Cut V_1 and V_2 along D_1 and D_2, push the new ends apart and take their closures to create two cylinders, C_1 and C_2, respectively. Denote the top of C_i by D_1', $i = 1, 2$, and the bottom by D_i''. We can define $h' : \partial C_1 - \text{int}(D_1' \cup D_1'') \to C_2 - \text{int}(D_2' \cup D_2'')$ to be compatible with h and glue C_1 to C_2. The boundary of $C_1 \cup C_2$ has two components: $S_1 = D_1' \cup D_2'$ and $S_2 = D_1'' \cup D_2''$. These have to be 2-spheres. Thus $C_1 \cup C_2 \simeq S^2 \times I$. We can recover $V_1 \cup_h V_2$ by gluing S_1 to S_2. The resulting manifold is homeomorphic to $S^2 \times S^1$.

We want to generalize this and classify all manifolds that can be constructed by gluing two solid tori together. The resulting spaces are call **lens spaces**. More detailed treatments can be found in [15, 18]. Let $h : \partial V_1 \to \partial V_2$ be a homeomorphism from the boundary of solid torus V_1 to the boundary of solid torus V_2. Let $\mathcal{L} = V_1 \cup_h V_2$. Select a meridian-longitude pair for each and call them (L_i, M_i), for $i = 1, 2$ respectively. Notice we did not say the longitudes were standard. That's because these are abstract tori; they are not embedded in a larger space so we cannot talk about a curve bounding an outside disk – there is no outside yet! Nonetheless, we can still use them for create coordinate systems.

Suppose $h(M_1)$ is a (p, q)-curve on V_2. Let D_1 be a disk in V_1 with boundary M_1 and thicken it up a little bit to get a ball $B_1 \simeq D_1 \times I$ within V_1. The complement of the of B_1 in V_1 is another ball whose closure we denote B_1'. Now $B_2 = h(B_1)$ is a ball in $\mathcal{L}$. Consider the space $B_2 \cup_h V_2$; it is $\mathcal{L}$ with the interior of $B_2' = h(B_1')$ removed. Thus the topology of $\mathcal{L}$ does not depend on how B_2' is glued in. Hence $\mathcal{L}$ is determined by $B_2 \cup_h V_2$ which only depends on the (p, q) curve. Thus we define the (p, q)-lens space, also denoted $\mathcal{L}(p, q)$, to be the result of gluing a solid torus V_1 to V_2 with a meridian of V_1 going to a (p, q) curve on V_2.

We have that $\mathcal{L}(1, 0) \simeq S^3$ and $\mathcal{L}(0, 1) \simeq S^2 \times S^1$. Note that for $n > 1$ $\mathcal{L}(n, 0)$ and $\mathcal{L}(0, n)$ are not defined. Neither is $\mathcal{L}(0, 0)$.

Exercise 2.7. Convince yourself that $\mathcal{L}(1, q) \simeq S^3$ for all q.

While (p, q) determines $\mathcal{L}(p, q)$ the relationship is not unique. First, it can be shown that changing the signs of p or q does not affect the topology. It can be show that if $\mathcal{L}(p_1, q_1)$ is homeomorphic to $\mathcal{L}(p_2, q_2)$ we must have $p_1 = \pm p_2$. It is also known that $\mathcal{L}(p, q) \simeq \mathcal{L}(p, q + np)$ for all integers n. Thus we may assume $0 < q < p$ except for the cases $\mathcal{L}(0, 1)$ and $\mathcal{L}(1, 0)$. There is an additional symmetry and it turns out for $0 < q_1 < p$ and $0 < q_2 < p$, with p relatively prime to each q_i, is it known that $\mathcal{L}(p, q_1) \simeq \mathcal{L}(p, q_2)$ if and only if $\pm q_1 q_2^{\pm 1} = 1 \bmod p_1$.

Exercise 2.8. In Table 5.1 we have listed the lens spaces for $p = 2$ to 15 and given each equivalence class for a given value of p a different color. Verify it is correct and compute the next two rows. Hint: I wrote a Maple script to test for equivalence.

Exercise 2.9. The space formed by identifying opposite points on the boundary of a 3-ball is called the real projective space of dimension three and is denoted $\mathbb{R}P^3$ or just P^3. Show that $\mathcal{L}(2, 1) \simeq P^3$.

Exercise 2.10. Show that $S^2 \times S^1$ contains an embedded Klein bottle. Find a lens space that contains an embedded projective plane, P^2.

$\mathcal{L}(2,1)$											
$\mathcal{L}(3,1)$	$\mathcal{L}(3,2)$										
$\mathcal{L}(4,1)$	$\mathcal{L}(4,3)$										
$\mathcal{L}(5,1)$	$\mathcal{L}(5,2)$	$\mathcal{L}(5,3)$	$\mathcal{L}(5,4)$								
$\mathcal{L}(6,1)$	$\mathcal{L}(6,5)$										
$\mathcal{L}(7,1)$	$\mathcal{L}(7,2)$	$\mathcal{L}(7,3)$	$\mathcal{L}(7,4)$	$\mathcal{L}(7,5)$	$\mathcal{L}(7,6)$						
$\mathcal{L}(8,1)$	$\mathcal{L}(8,3)$	$\mathcal{L}(8,5)$	$\mathcal{L}(8,7)$								
$\mathcal{L}(9,1)$	$\mathcal{L}(9,2)$	$\mathcal{L}(9,4)$	$\mathcal{L}(9,5)$	$\mathcal{L}(9,7)$	$\mathcal{L}(9,8)$						
$\mathcal{L}(10,1)$	$\mathcal{L}(10,3)$	$\mathcal{L}(10,7)$	$\mathcal{L}(10,9)$								
$\mathcal{L}(11,1)$	$\mathcal{L}(11,2)$	$\mathcal{L}(11,3)$	$\mathcal{L}(11,4)$	$\mathcal{L}(11,5)$	$\mathcal{L}(11,6)$	$\mathcal{L}(11,7)$	$\mathcal{L}(11,8)$	$\mathcal{L}(11,9)$	$\mathcal{L}(11,10)$		
$\mathcal{L}(12,1)$	$\mathcal{L}(12,5)$	$\mathcal{L}(12,7)$	$\mathcal{L}(12,11)$								
$\mathcal{L}(13,1)$	$\mathcal{L}(13,2)$	$\mathcal{L}(13,3)$	$\mathcal{L}(13,4)$	$\mathcal{L}(13,5)$	$\mathcal{L}(13,6)$	$\mathcal{L}(13,7)$	$\mathcal{L}(13,8)$	$\mathcal{L}(13,9)$	$\mathcal{L}(13,10)$	$\mathcal{L}(13,11)$	$\mathcal{L}(13,12)$
$\mathcal{L}(14,1)$	$\mathcal{L}(14,3)$	$\mathcal{L}(14,5)$	$\mathcal{L}(14,10)$	$\mathcal{L}(14,11)$	$\mathcal{L}(14,13)$						
$\mathcal{L}(15,1)$	$\mathcal{L}(15,2)$	$\mathcal{L}(15,4)$	$\mathcal{L}(15,7)$	$\mathcal{L}(15,8)$	$\mathcal{L}(15,11)$	$\mathcal{L}(15,13)$	$\mathcal{L}(15,14)$				

Table 2.1: Lens spaces, $p = 2$ to 15, colored by topological type

Remark. In most references $\mathcal{L}(0,1)$ and $\mathcal{L}(1,0)$ are not accepted as lens spaces. This makes stating some theorems cleaner.

Remark. Lens spaces are prime.

Chapter 3

Homology

The idea in *algebraic topology* is that each topological space can be associated to some algebraic object like a group. If two given spaces have non isomorphic associated groups then they cannot be homeomorphic. Thus a topological problem is reduced to an algebraic one. Here we develop the **first homology group** of a manifold. It can be defined for any space but we will be working with manifolds. If G_1 and G_2 are isomorphic groups we will write $G_1 \equiv G_2$.

1 From simplices to groups

Let $O = \{(0,0)\}$, $I = [0,1] \times \{0\}$ and let T be the triangle shaped disk in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$ and $(1,0)$. Given a manifold M we define three groups. Consider first the set S_0 of all maps from O into M. These are called **zero dimensional simplices** or **0-simplices** and can be thought of as just the points of M. Then let C_0 be all formal sums of the form $n_1p_1 + \cdots + n_kp_k$ where each $p_i \in S_0$ and each $n_i \in \mathbb{Z}$. These are called the **0-chains** of M. We add elements of C_0 is the obvious way and let 0 stand for the null symbol. For example,

$$(2p_1 + 3p_2 - p_3) + (p_1 - 3p_2 + 2p_4) = 3p_1 + 0p2 - p_3 + 2p_4 = 3p_1 - p_3 + 2p_4.$$

With this C_0 becomes a group. We also declare that $P + Q = Q + P$ for any two members of C_0 so that C_0 is Abelian. We call C_0 the **0-chain group**.

Next let S_1 be the set of all continuous maps of I into M. These are called **1-simplices**. We define C_1, the **1-chain group**, just as we did the 0-chain group by using linear combinations of 1-simplices with integer coefficients.

Finally define the set of **2-simplices**, S_2 as, you guessed it, the set of all continuous maps from T into M and then define **2-chain group** C_2 as formal linear combinations of 2-simplices with integer coefficients.

In each case we have n-simplices generating n-chains forming the n-chain group.

The three chains groups, by themselves, do not tell us much about the manifold, but the interactions between them will. We define two group homomorphisms $\partial_1 : C_1 \to C_0$ and $\partial_2 : C_2 \to C_1$, called the **boundary maps**, as follows. Let $s \in S_1$. Define $\partial_1(s) = s(1) - s(0)$, that is the two end points thought of as a 0-chain. We can linearly extend this set map from S_1 into C_0 to a homomorphism from C_1 into C_0. For example, let u and v be in S_1. Then

$$\partial_1(2u + 3v) = 2u(1) - 2u(0) + 3v(1) - 3v(0).$$

Why the minus sign? Suppose u and v are such that $u(1) = v(0)$. Then $\partial(u + v) = v(1) - u(0)$ which seems "natural". Why? We also impose the following, that for any 1-simplex $u(t)$ we set the map $t \to u(1-t)$ to be $-u$.

Defining ∂_2 is a little harder. The disk T has three line segments that make up its boundary. Each can be thought of as a copy of I. Let $z \in S_2$. We can think of each restriction of z to a boundary segment of T as a 1-simplex. Our definition of $\partial_2 z$ will be some linear combination of the three 1-simplices defined by restricting z to each edge segment of T. The coefficients will be ± 1. To get the signs "right" will take a little effort. First we represent z by the image of T's vertices in order. Suppose $z(0,0) = a$, $z(1,0) = b$ and $z(0,1) = c$. Then we would write $[abc]$ for z. Obviously many other 2-simplices would have the same symbol; this will not hurt anything. Now for the 1-simplex given by z restricted to the edge from $(0,0)$ to $(1,0)$ write $[ab]$. Define $[bc]$ and $[ac]$ similarly. Then we define $\partial_2 z = \partial_2[abc] = [bc] - [ac] + [ab]$. Now watch.

$$\partial_1\partial_2(z) = \partial_1([bc] - [ac] + [ab]) = (c - b) - (c - a) + (b - a) = 0.$$

So, the boundary of the boundary is empty! This is what we wanted. Now extend ∂_2 linearly to get a homomorphism from C_2 into C_1.

Since $\partial_1\partial_2 c = 0$ for any $c \in C_2$ (convince yourself of this) we know the image of ∂_2 is contained in the kernel of ∂_1; in fact it is a subgroup. The kernel of ∂_1 is called the subgroup of **1-cycles**. The image of ∂_2 is called the subgroup of **1-boundaries**, that is they are 1-cycles that are the boundary of some 2-chain. The **first homology group** of a manifold M is defined to be

$$H_1(M) = \frac{\text{kernel}\,\partial_1}{\text{image}\,\partial_2}.$$

It contains a wealth of information about M. One can go on to define the n-chain groups and boundary maps $\partial_n : C_n \to C_{n-1}$ for any n and then define the n-th homology groups by $H_n(M) = \text{kernel}\,\partial_n/\text{image}\,\partial_{n+1}$. We will only need $H_1(M)$.

If c_1 and c_2 are 1-cycles in the same homology class, this means there exist a two 2-chain d such that $\partial_2 d = c_1 - c_2$, we say that c_1 is **homologous** to c_2 and write $c_1 \sim c_2$. If c_1 is in the trivial equivalence class, that is it is in the identity element of H_1, then we say c_1 is **null homologous**.

For 1-simplices it is easy to that $[ab] = -[ba]$. Switching the vertices a and b changes the orientation of the 1-simplex. What happens when we reorder the vertices of $[abc]$? Observe that

$$\partial_2[abc] = [bc] - [ac] + [ab] = [ab] + [bc] + [ca]$$

while

$$\partial_2[bac] = [ac] - [bc] + [ba] = -[ab] - [bc] - [ca] = -\partial_2[abc].$$

Since it is useful to set $[abc] = -[bac]$. In general any time we switch two adjacent letters the sign changes. We can think of the underlying triangle as having one of two orientations, clockwise or counterclockwise. In Figure 3.1 $[abc]$ is ccw, while $[bac]$ is cw.

Exercise 3.1. The check that switching any two adjacent vertices changes the sign and that

$$\partial_2[abc] = \partial_2[cab] = \partial_2[bca] = -\partial_2[acb] = -\partial_2[bac] = -\partial_2[cba].$$

Exercise 3.2. Consider the square shown in Figure 3.2 which is covered by a two 2-simplices, T_1 and T_2. If both have the same orientation show that $\partial(T_1 + T_2)$ is a 1-cycle covering the boundary of the square, but if they have different orientations this is not true.

Next we find the first homology groups of some of the manifolds we have looked at. We won't be rigorous here. I want to focus on the intuition for

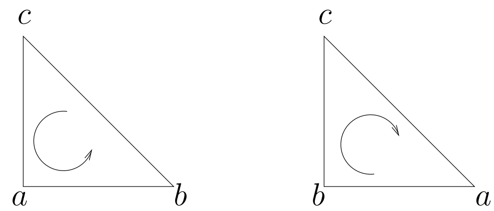

Figure 3.1: Left: $[abc]$. Right: $[bac]$.

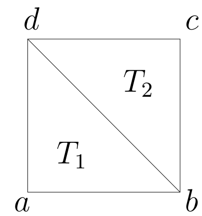

Figure 3.2: Square

simplicity and because it is the basic intuitive idea that led to the definition of the homology groups. Start with the disk D^2. Draw some segments that form a cycle with no self crossings. Notice they are the boundary for a smaller disk. Thus in $H_1(D^2)$ that 1-cycle is in the same equivalence class as the identity. Cycles in D^2 can be very complicated and have many self crossings. But the algebra works out such that they are all the boundary of some 2-chain. Thus, $H_1(D^2) = 0$, the trivial group with only the identity element. For the 2-sphere we get the same result. So far homology is not too interesting!

Now consider an annulus A. It is easy to draw a 1-cycle that goes around the hole and thus does not bound any 2-chain; see Figure 3.3. Therefore $H_1(A)$ is not the trivial group. If we draw two non-intersecting 1-cycles with no self crossings around the hole then together they form the boundary of a smaller annulus which is the image of some 2-chain. Thus these two 1-cycles are in the same homology class. (We are glossing over some orientation issues.) Consider the 1-cycle formed by taking one of the 1-cycles in Figure 3.3 and assigning a weight of 5 to each of the 1-simplices. We can think of it as representing walking around the annulus five times. It still has boundary 0 and there is no 2-chain whose boundary it could be. It turns out the homology class of any 1-cycle that "goes around once" generates $H_1(A)$ and that $H_1(A) \equiv \mathbb{Z}$.

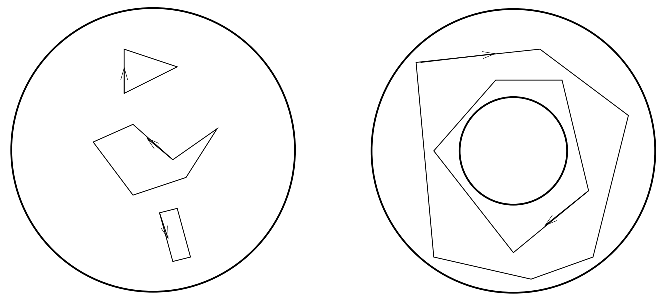

Figure 3.3: 1-cycles in a disk and an annulus

If we take a disk and cut out two disjoint smaller disks then $H_1 \equiv \mathbb{Z}^2$, and so on. Thus H_1 is a kind of hole counter. Now consider the torus T^2. How many holes does it have? It might seem like it has only one, but in a sense it has two. The donut hole in the middle but also the "hollowed out" interior. It turns out we can generate the homology group with two classes: the equivalence class of a 1-cycle going around the "donut hole" is one and a 1-chain going around meridian of the tube is the other. Thus, $H_1(T^2) \equiv \mathbb{Z}^2$.

Exercise 3.3. It turns out $H_1(S^1) \equiv \mathbb{Z}$. Justify this.

Exercise 3.4 (Hard)**.** When we punched a hole in the disk the homology group changed from trivial to one isomorphic to $\mathbb{Z}$. This is because certain 1-cycles that where 1-boundaries in the disk where no longer 1-boundaries in the annulus. Now, suppose we punch a hole in a torus. Will the homology group change? Are there 1-boundaries that no longer bound a 2-chain? What if we punch out two holes?

Exercise 3.5. For a manifold M, $H_0(M)$ is defined to be $C_0/\text{image } \partial C_1$. Convince yourself that if M has n path connected components then $H_0(M) \equiv \mathbb{Z}^{n-1}$.

2 Induced homomorphisms

Let $f : M \to N$ be continuous. Let $[A] \in H_1(M)$ where $A = \sum n_i s_i$ with the s_i's being 1-simplices and the n_i's integers. We define $f_*([A]) = [\sum n_i f \circ s_i]$.

This determines a function $f_* : H_1(M) \to H_1(N)$. It can be shown that f_* is a group homomorphism and it is called the *induced homomorphism* of f. If f is a homeomorphism then f_* is an isomorphism. But there are non-homeomorphic spaces with isomorphic homology groups.

Example 3.1. We can now prove that S^2 and T^2 are not homeomorphic and that neither is homeomorphic to S^1.

3 Homotopy

Definition 3.1. Two continuous functions $f_i : M \to N$ for $i = 0, 1$ are **homotopic** if there exists a continuous function $H : M \times I \to N$ such that for all $x \in M$

1. $H(x, 0) = f_0(x)$,
2. $H(x, 1) = f_1(x)$.

Let $f : M \to \mathbb{R}^n$, be continuous where M is any compact space. Then f is homotopic to the map $z : M \to \{0\} \subset \mathbb{R}^n$. There is a way to paste homotopies together and using this one can show any two continuous functions from M into $\mathbb{R}^n$ are homotopic. The same is true if $\mathbb{R}^n$ is replaced by S^n or D^n.

Example 3.2. Let C be a compact topological space and $f : C \to \mathbb{R}^n$ be continuous. Let $H : C \times I \to \mathbb{R}^2$ be given by $H(x, t) = f(x) \cdot (1 - t)$. Then $H(x, 0) = f(x)$ and $H(x, 1) = (0, 0)$.

But if we consider functions from S^1 into an annulus this is not true. It can be shown that two maps from S^1 into an annulus are homotopic if and only if they have the same winding number with respect to the hole. (It is possible to use this idea to define *homotopy groups*. It the case of an annulus the homotopy group and the homology group are isomorphic; this is not true in general.)

It is possible to extend the idea of homotopy to chains. Roughly speaking, we define two n-chains to be homotopic if there if a homotopy between the underlying sets and that respects the weight on each simplex. It can be shown that if two n-cycles are homotopic they are also homologous. This is intuitively plausible but technically messy to prove.

Definition 3.2. Let X and Y be topological spaces. If there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to identity maps then we say X and Y are *homotopic* spaces.

Any two spaces that are homeomorphic are obviously homotopic. The classification of topological spaces up to homotopy equivalence is coarser than for topological equivalence.

Example 3.3. Let U be the unit circle on $\mathbb{R}^2$ given the subspace topology. Let $U' = U \cup [1,2] \times \{0\}$ given the subspace topology from $\mathbb{R}^2$. Then U and U' are homotopic spaces.

Example 3.4. Let T be the subspace of $\mathbb{R}^2$ given by $[-1,1] \times \{0\} \cup \{0\} \times [0,1]$. We will show that T and $[-1,1]$ are not homeomorphic but are homotopic.

Proof. We can use cut point theory to show they are not homeomorphic. To show they are homotopic let $H((x,y),t) = (x, y \cdot (1-t))$. □

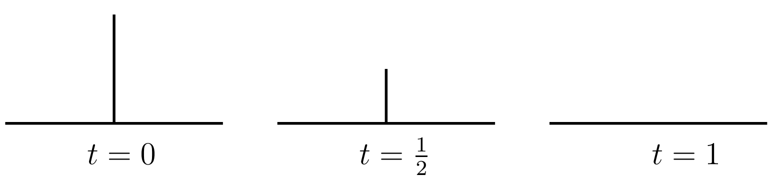

Figure 3.4: Homotopy for Example 3.4

The following theorem, which is proved in courses on algebraic topology, is of major importance.

Theorem 3.1. *Homotopic spaces have isomorphic homology groups.*

Thus I, S^1, and T^2 are not homotopic to each other. However, there are homotopies between an annulus and S^1, between T^2 minus a point and the wedge of two circles (a "figure 8"), between I, D^2 and a point. A space homotopic to a point is said to be **retractable**.

Exercise 3.6. Construct homotopies for the cases discussed above.

Example 3.5. Let $\ddot{M}$ denote the Möbius band. It is homotopic to its core which is a copy of S^1. Therefore $H_1(\ddot{M}) \equiv H_1(S^1) \equiv \mathbb{Z}$. A 1-cycle, C, running along the core could not be the boundary of a 2-chain. Its equivalence class generates $H_1(\ddot{M})$. But you might think a 1-cycle that traces along the boundary of $\ddot{M}$ would be the boundary of a 2-chain that covers the Möbius band and would thus be null homologous. But this is not the case. Consider the 2-chain shown in Figure 3.5. It covers $\ddot{M}$ and each 2-simplex has coefficient 1. But no matter how we orient the 2-simplices the boundary of their sum does not give a 1-cycle that covers $\partial\ddot{M}$.

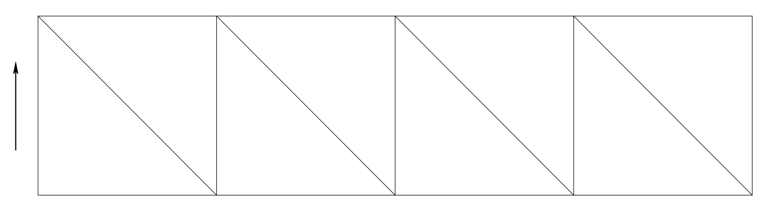

Figure 3.5: Möbius band

Exercise 3.7. Prove that it is impossible to orient the 2-simplices in Figure 3.5 so that the boundary of the 2-chain formed by there sum coincides with the boundary of the Möbius band.

The problem is that $\ddot{M}$ is unorientable; a 1-cycle that covers once the boundary of an *orientable* surface will be the boundary of some 2-chain and hence be null homologous.

Example 3.6. Recall the construction of the projective plane, glue a disk along its boundary to the boundary of a Möbius band. Notice that now a 1-cycle, call it B, covering the boundary of the Möbius band bounds a disk - which is certainly an orientable surface. Thus B is null homologous. Since B is homologous to $2C$, where C is a cycle along the core of the Möbius band, we have $2C \sim 0$. This can be used to show that

$$H_1(P^2) \equiv \mathbb{Z}/2\mathbb{Z}.$$

We can no longer think of H_1 as simply a hole counter since in this case it is picking up more subtle information about the manifold.

We state two useful facts.

Fact. The homology groups of surfaces are $H_1(F_n) \equiv \mathbb{Z}^{2n}$ and $H_1(G_n) \equiv \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2\mathbb{Z}$. If we remove m disks from a surface without boundary the effect on H_1 is to increase the exponent of $\mathbb{Z}$ by $m-1$. (Think about why this is.)

Fact. For lens space it is known that $H_1(\mathcal{L}(p,q)) \equiv \mathbb{Z}/p\mathbb{Z}$. It follows that if $\mathcal{L}(p_1, q_1)$ is homeomorphic to $\mathcal{L}(p_2, q_2)$ we must have $p_1 = \pm p_2$, as mentioned earlier. (Think about why this is.)

4 Finitely Generated Abelian Groups

The first homology group of any compact manifold is a finitely generated Abelian group. The name means just what it says. These are Abelian groups that can be generated by a finite number of elements. For example $\mathbb{Z}$ is generated by $\{1\}$. So is $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$, where the "1" means something different; it is the equivalence class of integers whose remainder upon division by 3 is 1. The subset $\{(1,0), (0,-1)\}$ of $\mathbb{Z}^2$ generates the group $\mathbb{Z}^2$ using vector addition.

Therefore we take a brief algebraic detour to study this class of groups. The set $\mathbb{Z}^n$ is an Abelian group under vector addition. It can be generated by a finite number of elements. (Give an example of a generating subset. Give another.) Let A be an $n \times n$ matrix with integer entries. It induces a function from $\mathbb{Z}^n$ to $\mathbb{Z}^n$ via matrix multiplication. We will denote this function by A so $A : \mathbb{Z}^n \to \mathbb{Z}^n$. By standard facts from linear algebra it is a homomorphism. The image of the homomorphism is a subgroup of $\mathbb{Z}^n$ which we denote by $A\mathbb{Z}^n$. For example multiplication by 2 from $\mathbb{Z}$ into $\mathbb{Z}$ has the even integers as its image. In our notation $2\mathbb{Z} = \{\dots, -2, 0, 2, \dots\}$. (We are writing the 1×1 matrix $[2]$ as 2.)

Given an Abelian group G and a subgroup H one can form the quotient group G/H. The members of G/H are subsets of G that differ by a member of H. For example, $\mathbb{Z}/2\mathbb{Z}$ has two elements, the even integers and the odd integers. The induced addition operation is that even plus even is even, odd plus odd is even and even plus odd is odd. We tend to be sloppy and write $\mathbb{Z}/2\mathbb{Z}$ as $\{0, 1\}$, taking addition to be addition mod 2, but really the 0 stands for the set of even integers and the 1 stands for the set of odd integers.

Let $G = \mathbb{Z}/n\mathbb{Z}$. If $n = \pm 1$ then $n\mathbb{Z} = \mathbb{Z}$ and G is the trivial group, which has one element. In general the number of elements in G, that is the **order** of G, is $|n|$, unless $n = 0$. In this case $0\mathbb{Z} = \{0\}$ and $G \equiv \mathbb{Z}$ and so has infinite order.

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $G = \mathbb{Z}^2/A\mathbb{Z}^2$. The reader should work out that $G \equiv \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ which has six elements. If instead $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ the reader should check that $G \equiv \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, which has infinitely may elements. In general we have the following theorem which we shall not prove.

Theorem 3.2. *Let A be an $n \times n$ integer matrix. Then the order of $\mathbb{Z}^n/A\mathbb{Z}^n$ is $|\det A|$ if $\det A$ is not zero and is infinite if $\det A = 0$.*

Quite a bit more is true. Any finitely generated Abelian group is isomorphic to $\mathbb{Z}^n/A\mathbb{Z}^n$ for some n and integer $n \times n$ matrix A. While it can happen that different matrices yield isomorphic groups there is a simple algorithm involving row and column operations that determines when this happens [4].

Exercise 3.8. a. Verify the claims made just before Theorem 3.2.

b. Show that the groups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ are isomorphic.

c. Show that the groups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ are not isomorphic.

d. Show that $\dfrac{\mathbb{Z}^2}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

e. Show that $\dfrac{\mathbb{Z}^2}{\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \mathbb{Z}^2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

Chapter 4

Torus Maps

It is possible and useful to represent a homeomorphism from a torus to itself or another torus by a 2×2 matrix.

Think of the torus T^2 as given by $I \times I$ with the opposite edges identified. A 2×2 integer matrix A induces a map from $\mathbb{R}^2$ to itself that preserves the integer lattice. If we use arithmetic modulo 1 then A determines a map from $[0,1) \times [0,1)$ to itself. From this we can get a map from T^2 to T^2.

Exercise 4.1. Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -4 \end{bmatrix}$. Then define $A(x,y) = (5x + y, 3x - 4y)$ mod 1. Thus, $A\left(\frac{1}{2}, \frac{1}{6}\right) = \left(\frac{2}{3}, \frac{5}{6}\right)$. However, this map is not one-to-one. Find another point $(x,y) \in [0,1) \times [0,1)$ such that $A(x,y) = (\frac{2}{3}, \frac{5}{6})$. In fact this map is 23-to-1! Find 23 points $(x,y) \in [0,1) \times [0,1)$ such that $A(x,y) = (0,0)$.

Fact. A 2×2 integer matrix A induces a homeomorphism on T^2 if and only if $\det(A) = \pm 1$.

Exercise 4.2. Prove this.

Exercise 4.3. a. Let $A = \begin{bmatrix} 2 & 1 \\ a & b \end{bmatrix}$ and suppose $\det(A) = \pm 1$. What are all the possible integer values for a and b?
b. Let $A = \begin{bmatrix} 2 & 4 \\ a & b \end{bmatrix}$. Are there any integer values for a and b such that $\det(A) = \pm 1$?

Exercise 4.4. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. In Figure 4.1 we show the image of $I \times I$ under the action of A in $\mathbb{R}^2$ and how it wraps around the torus when using mod 1 arithmetic. Redo this for $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Try to draw each of these on a donut!

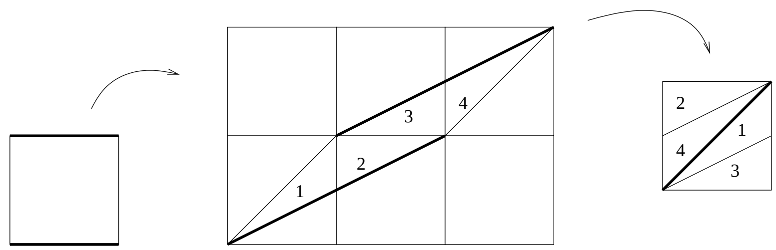

Figure 4.1: A linear torus homeomorphism

These maps are linear. Of course there are nonlinear self homeomorphisms of the torus. For our proposes, they don't matter since we shall see that any self homeomorphism of the torus is essentially equivalent to a linear one in the sense defined next.

Definition 4.1. Two homeomorphisms $f_i : M \to M$ for $i = 0, 1$ are **isotopic** if there exists a continuous function $H : M \times I \to M$ such that

1. $H(x, 0) = f_0(x)$,
2. $H(x, 1) = f_1(x)$, and
3. $H(x, t) : M \to M$ is a homeomorphism for each fixed t.

We state two theorems without proof, but they should be intuitively plausible. Proofs can be found in [1] and [20], respectively.

Theorem 4.1. *Every homeomorphism from a torus to a torus is isotopic to one induced by a* 2×2 *integer matrix with determinant* ± 1*. See Figure 4.2.*

Theorem 4.2. *Let* M *be a 3-manifold and let* T *be a torus component of* ∂M*. Let* V *be a solid torus and let* $f_i : \partial V \to T$ *for* $i = 0, 1$ *be homeomorphisms. Let* $M_i = M \cup_{f_i} V$ *be the result of gluing* V *to* M *via* f_i*. If* f_0 *is isotopic to* f_1*, then* M_0 *is homeomorphic to* M_1*.*

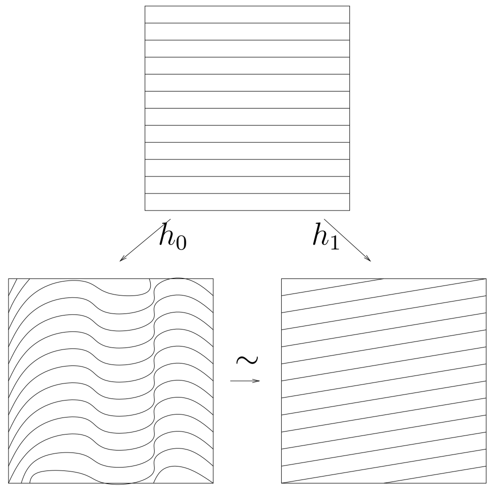

Figure 4.2: Isotopic Torus Homeomorphisms

These two theorems tell us that the topological type of a Dehn surgery is determined by the 2×2 matrix of a linearization of the attaching map. However, it turns out that only the first column of the matrix is needed. To see this we need to know a little more about what goes on inside a solid torus.

Exercise 4.5. Compare and contrast the definitions of ambient isotopy, isotopy and homotopy. How are they similar how are they different? Look their definitions up in several textbooks and compare.

Fact. Let $f : T^2 \to T^2$ be continuous. Write T^2 as $S^1 \times S^1$ and use the two S^1 factors for loops, with suitable orientations, whose equivalence classes generate $H_1(T^2)$. Then the linearization matrix of f is the same as the matrix for the group homomorphism f_*.

Chapter 5

Seifert Manifolds

A **Seifert fibered manifold** is a 3-manifold $\mathcal{M}$ together with a fiber structure $\mathcal{F}$ that is a decomposition of $\mathcal{M}$ into a union of disjoint copies of S^1, called the fibers. We will start by describing Seifert fibrations of the solid torus. The main references we have drawn on for this material are Seifert's original paper (in translation) [20], the course notes of Brin [2], a draft textbook by Hatcher [5].

Consider the solid cylinder $C = D^2 \times I$. Let $\mathcal{F}_c = \{\{(r, \theta)\} \times I : (r, \theta) \in D^2\}$. This gives a fibration of C by closed intervals. Let $D_i = D^2 \times \{i\}$ for $i = 0, 1$ be the bottom and top disks of C respectively. For any real number ψ let $R_\psi : D_0 \to D_1$ be given by $R_\psi(r, \theta, 0) = (r, \theta + \psi, 1)$. Identify D_0 and D_1 using R_ψ as the homeomorphism to form a solid torus V. See Figure 5.1. If ψ is a rational multiple of π the fibers of C become joined at their end points to form circles. The core circle will contain just one copy of I. If $\psi = \frac{m}{n} 2\pi$ for coprime integers m and n then the other circles will be formed from n copies of I. Such an object is called a $\left(\frac{\mathbf{m}}{\mathbf{n}}\right)$**-fibration of the solid torus** and we will denote it by $V_{m/n}$. The core is called the **exceptional fiber** unless $m = 0$ or $n = 1$ in which case we say the fibration is trivial. The **index** of an exceptional fiber is the number of times nearby fibers meet a meridianal disk and is equal to n. (Notice that for $n = 1$ it does not matter what m is; the identification map with be the identity map.) Non-exceptional fibers are called **ordinary** or **regular fibers**. If $V_{m/n}$ is standardly embedded in $\mathbb{R}^3$ then it is fibered by (n, m) torus knots and its core. Now we can give the general definition.

Definition 5.1. A **Seifert fibered manifold** is a 3-manifold $\mathcal{M}$ together with a fiber structure $\mathcal{F}$ that is a decomposition of $\mathcal{M}$ into a union of disjoint

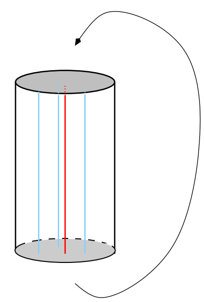

Figure 5.1: Fibered Cylinder/Solid Torus

copies of S^1, called the fibers, such that each S^1 fiber not in $\partial\mathcal{M}$ has a closed tubular neighborhood that is a Seifert fibered solid torus and if $\partial\mathcal{M} \neq \phi$ the boundary components of $\mathcal{M}$ are tori fibered as though each was the boundary of a Seifert fibered solid torus[1].

If $\mathcal{M}$ is compact it can be shown that the number of exceptional fibers is finite.

A given manifold $\mathcal{M}$ may have several *different* Seifert fibrations, only one or none. Manifolds that do not have Seifert fibrations include the 3-ball and all non prime manifolds except $P^3\#P^3$. But what do we mean by *different*? Two fibrations of a manifold $\mathcal{M}$ are **fiber equivalent** if there is a homeomorphism $h : \mathcal{M} \to \mathcal{M}$ that takes fibers to fibers. We will denote this by $\cong$. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two fibrations of a manifold $\mathcal{M}$. They are **fiber isotopic** if there is a continuous function $S : \mathcal{M} \times I \to \mathcal{M}$ such that

1. $S(\cdot, 0)$ is the identity on $\mathcal{M}$.

2. For any $t \in I$ the $S(\cdot, t)$ induces a fibration on $\mathcal{M}$. That is for each fiber $F \in \mathcal{F}_0$ the image $S(F, t)$ is a circle and for each t all these circles fit together to form a fibration.

3. Finally, for $t = 1$ the induced fibration is $\mathcal{F}_1$.

Isotopic fibrations are fiber equivalent.

[1]Some authors allow for Klein bottles in the boundary too.

1 Fibrations of the Solid Torus

Next we ask, when are two fibrations of the solid torus, $V_{m/n}$ and $V_{m'/n'}$, fiber equivalent? Changing the signs of both m and n makes no difference and changing the sign of only one of them gives a fibration that is the mirror image of the original. As we noted above full twists of the solid torus produce fiber equivalent fibrations. Thus, disregarding the trivial fibration, we might as well only allow $0 < m < n$. However we can make a further restriction. Since $V_{m/n} \cong V_{-m/n} \cong V_{\frac{-m}{n}+1} = V_{\frac{n-m}{n}}$ we can assume $m \leq n/2$. Since m and n must be coprime the only case where equality can occur is when $n = 2$, which implies $m = 1$.

Theorem 5.1. *Let $V_i = V_{m_i/n_i}$ for $i = 1, 2$ with $0 < m_i \leq n_i/2$, and suppose $h : V_1 \to V_2$ is a fiber preserving homeomorphism. Then $m_1 = m_2$ and $n_1 = n_2$.*

Proof. For $i = 1, 2$ let $F_i \subset \partial V_i$ be fibers and suppose $F_2 = h(F_1)$. We may choose meridian-longitude pairs (M_i, L_i) for V_i respectively and use them as generators of $H_1(\partial V_i)$. Then we have

$$F_i \sim m_i M_i + n_i L_i$$

for $i = 1, 2$.

Now let D_i be a meridianal disk in V_i with $D_2 = h(D_1)$. Then F_i meets D_i in n_i points. Since $F_2 = h(F_1)$ meets D_2 in n_1 points we have $n_1 = n_2$; call this number n.

We can restrict h to ∂V_1 and then recall that it induces of homomorphism from $H_1(\partial V_1)$ to $H_1(\partial V_2)$. To keep the notation simple we will also denote this function by h.

Notice that $h(M_1) \sim \pm M_2$. If need be switch the orientation of M_2 and L_2 so that $h(M_1) \sim M_2$ without affecting the value of m_2. Also $h(L_1)$ meets $h(M_1)$ in one point and hence is a longitude. Thus $h(L_1) \sim \pm L_2 + xM_2$ for some integer x. We compute

$$F_2 = h(F_1) \sim m_1 h(M_1) + nh(L_1) \sim m_1 M_2 + n(\pm L_2 + xM_2).$$

Hence

$$m_2 M_2 + nL_2 \sim m_1 M_2 \pm nL_2 + nxM_2,$$

giving

$$(m_2 - m_1 - nx)M_2 \sim (\pm n - n)L_2.$$

By linear independence both sides must be null homologous. Thus, $m_2 - m_1$ is a multiple of n. Because of the restriction $0 < m_i \leq n/2$ we have $m_1 = m_2$. □

Example 5.1. This is really a non-example. We give a decomposition of a solid torus into circles that is not a valid Seifert fibration. On D^2 use the concentric circles with center the origin. Then for $D^2 \times S^1$ let $\mathcal{F} = \{C \times \{\theta\} \,|\, \text{for each circle } C \text{ in } D^2 \text{ and } \theta \in S^1\} \cup \{\{0\} \times S^1\}$. This is not a Seifert fibration since there is no tubular neighborhood of $\{0\} \times S^1$ of the required type.

2 Fibrations of the 3-sphere

Suppose $V_i = V_{m_i/n_i}$ are glued via a homeomorphism $h : \partial V_1 \to \partial V_2$ that takes fibers to fibers and gives S^3. We know that if h is isotopic to the linear map

$$h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

the gluing will give S^3. Thus in this case

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ m_1 \end{bmatrix} = \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}.$$

Thus $m_2 = n_1$ and $n_2 = m_1$.

Figure 5.2 shows an example. One solid torus has a $2/3$ fibration and the other has a $3/2$ fibration. Thus there are two exceptional fibers. They are unknotted while all the other fibers are trefoil knots. We note that while $V_{3/2} \cong V_{1/2}$ such a fiber preserving homeomorphism cannot be extended over the rest of S^3.

If either or both m and n, are 1 we can have one or zero exceptional fibers respectively. It was proven by Seifert that these are the only fibrations of S^3 [20, §11].

3 Fibrations of the lens spaces

I may get to this later. It is very similar to the 3-sphere case.

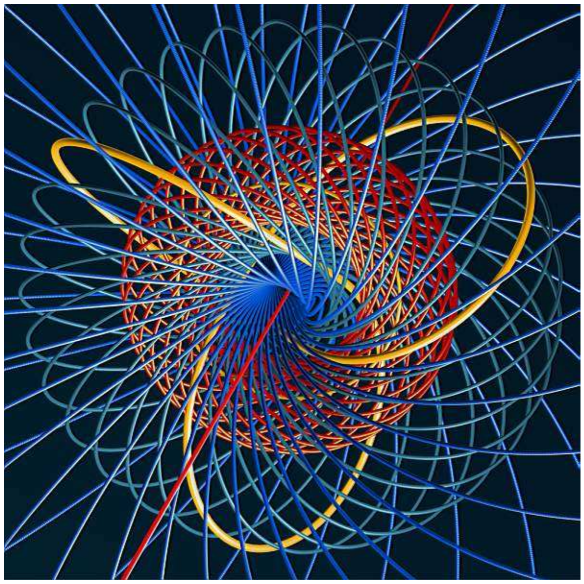

Figure 5.2: A Seifert fibration of S^3 [22]

4 Seifert fibered spaces over the 2-sphere

Additional examples of Seifert fibered spaces can be constructed as follows. Start with D^2 and create D_k by removing the interiors of k disjoint disks. Let $\mathcal{M}_0 = D_k \times S^1$. This is obviously a Seifert fibered space with $k+1$ tori as boundary components, T_i for $i = 0, ..., k$; the fibers are $p \times S^1$ for $p \in D_k$.

A **cross section** of a fibered space is an embedded surface that meets each fiber exactly once transversely. For $\mathcal{M}_0$ any surface $D_k \times *$, for some point $* \in S^1$, is a cross section. As we will see a little later there are many other possible choices.

Let V_i be solid tori and $h_i : \partial V_i \to T_i$ be homeomorphisms for $i = 0, ..., k$. Use the h_i's to glue in the V_i. Call the resulting manifold $\mathcal{M}$. Note that we have not as of yet placed a fibration on $\mathcal{M}$.

To characterize the h_i's we need to place a coordinate system on each ∂V_i and T_i. For each ∂V_i choose a meridian M_i and a longitude L_i. We will again regard M_i as having slope ∞ and L_i as having slope 0. On the T_i we don't have a clear notion of a meridian since T_i is not the boundary of a solid torus. We will keep the obvious fibration of $\mathcal{M}_0$ and use a fiber from it in T_i as the "slope ∞" axis. We will denote it F_i. For the "slope 0" axis we will select a cross section C of $\mathcal{M}_0$ and use its intersection with T_i as a coordinate axis of "slope 0". We will denote these Q_i and refer to them as **crossing curves**.

Now each h_i can be characterized, up to isotopy, by the slope of the image of M_i in T_i using the chosen coordinates. For each i let the slope be α_i/β_i. We require each slope to be finite, that is we do not allow h_i to send a meridian of ∂V_i to a fiber of T_i. We can then extend the fibration into each of the V_i. We now denote the *fibered* manifold by $S^2\left(\frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_k}{\beta_k}\right)$.

Example 5.2. Suppose that M_1 is taken by h_1 to a $\frac{5}{7}$ curve in T_1. Find a compatible fibration for V_1.

Solution. The matrix for h_* is on the form $\begin{bmatrix} a & 7 \\ b & 5 \end{bmatrix}$ since it must take $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} p \\ q \end{bmatrix}$. If we let $a = 4$ and $b = 3$ we get determinant -1. Then the fiber on ∂V_1 will be the inverse image of F_1. Thus,

$$\begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

Then V_1 has a $\frac{-4}{7}$ fibration. □

Exercise 5.1. Notice $S^2(\alpha/\beta) \simeq \mathcal{L}(\beta, \alpha)$. Show that $S^2(\alpha_0/\beta_0, \alpha_1/\beta_1) \simeq \mathcal{L}(\alpha_0\beta_1 + \alpha_1\beta_0, \alpha_0\alpha_1)$. This exercise is taken from [19].

A set of slopes, $\left\{\frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_k}{\beta_k}\right\}$, determines a Seifert fibered manifold. Of course the order they are given in does not matter and we might as well assume the α's and β's are coprime. But a space can be given by a different set of slopes.

It is easy to check that

$$S^2\left(\frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_k}{\beta_k}\right) \cong S^2\left(0, \frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_k}{\beta_k}\right).$$

We claim further that

$$S^2\left(\frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_i}{\beta_i}, \ldots, \frac{\alpha_j}{\beta_j}, \ldots, \frac{\alpha_k}{\beta_k}\right) \cong$$

$$S^2\left(\frac{\alpha_0}{\beta_0}, \ldots, \left(\frac{\alpha_i}{\beta_i}\right) + 1, \ldots, \left(\frac{\alpha_j}{\beta_j}\right) - 1, \ldots, \frac{\alpha_k}{\beta_k}\right).$$

The argument comes down to choosing a different cross section of $\mathcal{M}_0$ when determining the Q_i's. Start with the "obvious" cross section $D_k \times *$. Draw an arc A between any two boundary components of D_k. Then $A \times S^1$ is an annulus in $\mathcal{M}_0$. Cut $D_k \times *$ along A then bend one side of the slit up and the other down. Then reconnect the two sides of the silt with an annulus that wraps around $A \times S^1$ some number of times, say n. This creates a new cross section and changes the crossing curves on the two affected boundary tori of $\mathcal{M}_0$. See Figure 5.3. Then the slopes change by $+n$ on one of the tori and $-n$ on the other. If the arc went from a boundary curve back to itself the changes cancel out. It can be shown that, up to isotopy, all possible cross sections are obtainable in this way. Since changing the choice of cross section does not change to fibration we have proved the claim.

Figure 5.3: New Cross Section

We can now put the slopes into a normal form

$$S^2\left(\alpha_0, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_k}{\beta_k}\right)$$

where $0 < \alpha_i < \beta_i$ for $i = 1, \ldots, k$. The integer α_0 is called the **obstruction term**[2] If it is zero, it can be dropped. The other ratios we shall call the **gluing slopes**. The subscript k is the number of exceptional fibers.

[2]The name comes from a theorem showing that unless $\alpha_0 = 0$ even when the exception fibers are "drilled out" the orbit surface is not embedded in $\mathcal{M}_0 \cup V_0$. So, it is an obstruction to a certain embedding.

We can get many other examples, in fact all other examples, by starting with surfaces besides S^2 although some additional "twisting" is also involved.

One more bit of terminology. If we take any Seifert fibered space and then collapse each fiber to a point (look up *quotient* topology [10, 14]) the result is a surface. It is called the **orbit surface**. In all the examples we have studied the orbit surface is S^2 or, in the case of a solid fibered torus, a disk. The process of drilling out a tubular neighborhood of a fiber (ordinary or exceptional) and replacing with another fibered torus does not change the orbit surface. Note that the orbit surface is not usually a cross section of the whole manifold.

5 Homology groups of Seifert fibered spaces over the 2-sphere

Theorem 5.2. *Consider the Seifert fibered space $\mathcal{M}$ with orbit surface S^2 and k exceptional fibers with gluing slopes α_i/β_i for $i = 1, \ldots, k$ and obstruction term α_0. Then $H_1(\mathcal{M})$ is the Abelian group with $k+1$ generators, which we denote $F, Q_1, \ldots, Q_k$, and the $k+1$ relations,*

$$\alpha_0 F - Q_1 - \cdots - Q_k = 0 \text{ and } \beta_i F + \alpha_i Q_i = 0,$$

for $i = 1, \ldots, k$. The F generator can be represented by any ordinary fiber and the Q_i's can be represented by the crossing curves described above.

Sketch of Proof. We shall give only a rough justification. The homology group of a disk with k holes is isomorphic to $\mathbb{Z}^k$ and is generated by the boundary curves of the holes. If we take the cross product with S^1 we add a new generator; it turns out, no new relations are created. Thus $H_1(\mathcal{M}_0)$ is isomorphic to $\mathbb{Z}^{k+1}$. As we glue in solid tori new relations are created as crossing curves in the boundary tori are identified with boundaries of meridianal disks. We can choose to glue the solid tori with exceptional fibers to the inside tori giving the relations: $\beta_i F + \alpha_i Q_i = 0$ for $i = 1, \ldots, k$. For the outer boundary torus the crossing curve is homologous to $Q_1 + Q_2 + \cdots + Q_k$ (since together they bound a cross section surface) giving the remaining relation. □

Corollary 5.3. *For $n = 3$ the order of $H_1(\mathcal{M})$ is*

$$\pm(\alpha_0\beta_1\beta_2\beta_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3).$$

By choosing the orientation of $\mathcal{M}$ we can arrange it so the $+$ sign is used.

Proof. The relations can presented in matrix from as

$$\begin{bmatrix} \alpha_0 & -1 & -1 & -1 \\ \alpha_1 & \beta_1 & 0 & 0 \\ \alpha_2 & 0 & \beta_2 & 0 \\ \alpha_3 & 0 & 0 & \beta_3 \end{bmatrix}.$$

By Theorem 3.2 we just have to compute its determinant. □

Exercise 5.2. If there are no exceptional fibers then $S^2(\alpha_0)$ is a lens space. Which one?

Chapter 6

Trefoil Surgery Theorem

1 Dehn Surgery

Let $\mathcal{M}$ be a 3-manifold and let K be a knot in $\mathcal{M}$ with tubular neighborhood $N(K)$. Now remove the interior of $N(K)$ from M; let $\mathcal{M}_K = \mathcal{M} - \text{int } N(K)$. Let V be a solid torus disjoint from $\mathcal{M}$. Let $h : \partial V \to \partial N(K) \subset \mathcal{M}_K$ be a homeomorphism. Now glue V to $\mathcal{M}_K$ by using h to identify ∂V with $\partial N(K)$. Let $\mathcal{M}_{K,h} = \mathcal{M}_K \cup_h V$ denote the new manifold. Its topological type depends on both K and h but not on the choice of the tubular neighborhood. It is known that every compact orientable 3-manifold without boundary can be constructed via Dehn surgeries on S^3; this result is due to Lickorish and Wallace [18, Section 9.I.].

On $\partial N(K)$ choose a meridian and a preferred longitude, one that bounds a surface in $\mathcal{M}_K$. As with lens spaces the topological type of the resulting space is determined by which (p, q)-curve on $\partial N(K)$ a meridian of V is mapped to. Thus we talk about (p, q)-surgery on a knot K in S^3.

Fact. Let K be a knot in S^3. The manifold obtained from preforming a (p, q)-surgery on K has $H_1 \equiv \mathbb{Z}/q\mathbb{Z}$.

Explanation. Let K be a knot in S^3 and let $N(K)$ be a tubular neighborhood of K. Now let $\mathcal{M}_K = S^3 - \text{int } (N(K))$. Then $H_1(\mathcal{M}_K) \equiv \mathbb{Z}$.

We will only sketch the proof. The homology group of the boundary of $N(K)$ has two generators. We take these to be a meridian M of $N(K)$ and the preferred longitude L. Now L bounds a Seifert surface and hence is null homotopic, $L \sim 0$. That leaves only M and it can be shown no power of M is homologous to 0. Thus $H(\mathcal{M}_K) \equiv \mathbb{Z}$.

Take any knot complement manifold and glue in a solid torus using (p, q) surgery. Call the manifold $\mathcal{M}$. Now $|q|$ times the meridian of the knot is homologous to 0. The solid torus core is homologous to a preferred longitude which in turn is homologous to 0. These two facts can be used to show that $H_1(\mathcal{M}) \equiv \mathbb{Z}/q\mathbb{Z}$. ☐

2 Louise Moser's Theorem

We now have the tools in place to prove a classical theorem due to Louse Moser that tells us which three manifolds may result from surgery along a torus knot.

Theorem 6.1. *Let K be an (r, s) torus knot in S^3 and let $\mathcal{M}$ be the manifold that results from preforming a (p, q) Dehn surgery along K. Set $\sigma = rsp - q$.*

1. *If $|\sigma| > 1$ then $\mathcal{M}$ is a Seifert manifold over S^2 with three exceptional fibers of multiplicities $\beta_1 = s$, $\beta_2 = r$ and $\beta_3 = |\sigma|$. The proof will show how to compute the obstruction term and the three other α_i terms.*
2. *If $\sigma = \pm 1$ then $\mathcal{M}$ is the lens space $\mathcal{L}(|q|, ps^2)$.*
3. *If $\sigma = 0$ then $\mathcal{M}$ is $\mathcal{L}(r, s)\#\mathcal{L}(s, r)$.*

Proof. The Set Up. We will use the $\mathbb{R}^3 \cup \infty$ model for S^3. Let U be the unit circle in the xy-plane and let Z be the z-axis union $\{\infty\}$. We first partition S^3 into two solid tori, V_1' and V_2', with common boundary, where the core of V_1' is U and the core of V_2' is Z. Let M_i' and L_i' be preferred meridian-longitude pairs for V_i', $i = 1, 2$, where $M_1' = L_2'$ and $L_1' = M_2'$.

Now let K be an (r, s) torus knot on $\partial V_1' = \partial V_2'$. Let $N(K)$ be a tubular neighborhood of K that is small enough that $V_i = V_i' - \text{int}N(K)$ are still solid tori, $i = 1, 2$. Thus $V_1 \cup V_2$ is the knot complement space of K. The V_i look like the V_i' but with a trough dug out along K.

The intersection $\partial V_i' \cap \partial N(K)$ consists of two curves parallel to K. Call them K_i, $i = 1, 2$. They partition the boundary of each V_i into two annuli. Let A be the annulus between the K_i that the V_i have in common, that is $A = V_1 \cap V_2$. Let A_1 be $\partial V_1 - \text{int}A$ and A_2 be $\partial V_2 - \text{int}A$, that is A_1 and A_2 are the "bottoms" of the troughs.

Let (M_i, L_i) be meridian-longitude pairs for V_i, $i = 1, 2$ chosen by retracting M_i' and and L_i' through $N(K)$ as shown in Figure 6.1.

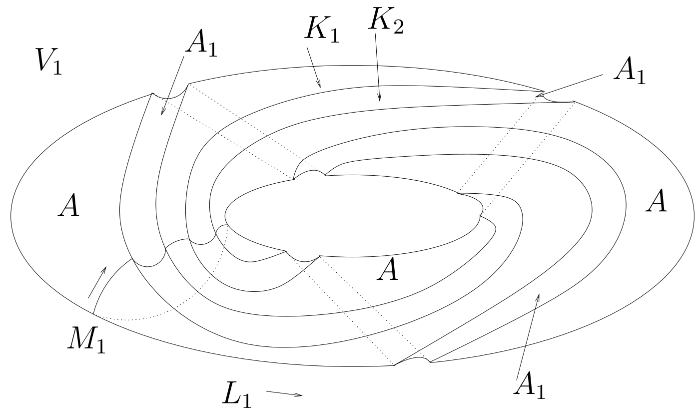

Figure 6.1: The Set Up: $\partial V_1 = A \cup A_1$; $\partial A = \partial A_1 = K_1 \cup K_2$

Let (M_3, L_3) be a preferred meridian-longitude pair for $N(K)$. Recall this means $L_3 \sim 0$ in $V_1 \cup V_2$.

Next let V_4 be a new solid torus with meridian-longitude pair (M_4, L_4). This is the solid torus we shall glue to $V_1 \cup V_2$ via a homeomorphism,

$$h : \partial V_4 \to \partial(V_1 \cup V_2) = \partial N(K).$$

Let $\begin{bmatrix} a & p \\ b & q \end{bmatrix}$ be the matrix representing h. Thus $h(M_4) = pL_3 + qM_3$.

The following homology calculation, done with respect to $V_1 \cup V_2$, will be used repeatedly. $K_1 \sim rZ$ and $Z \sim sM_3$ so $K_1 \sim rsM_3$. Thus $K_1 - rsM_3 \sim 0 \sim L_3$.

$$\begin{aligned} h(M_4) &= pL_3 + qM_3 \\ &\sim p(K_1 - rsM_3) + qM_3 \\ &= pK_1 - (rsp - q)M_3 \\ &= pK_1 - \sigma M_3. \qquad (*) \end{aligned}$$

Case 1. Suppose $|\sigma| \geq 2$. We augment our set up by using an (r, s) fibration of S^3 such that the knot K is now a fiber and the neighborhood of K is required to be a fibered neighborhood. In this fibration U and Z have multiplicities s and r respectively. We will need to figure out the fibration of V_4 such that h preserves fibers. The orbit surface will remain S^2 for the surgered manifold.

Now M_3 is a crossing curve on $\partial N(K)$ and K_1 is a fiber. Therefore, the fibration of V_4 will have a fiber of multiplicity $|\sigma|$ as its core. So we have a Seifert fibered space of the form $S^2\left(\alpha_0, \frac{\alpha_1}{s}, \frac{\alpha_2}{r}, \frac{\alpha_3}{|\sigma|}\right)$.

Example 6.1. Suppose K is a $(3,2)$ torus knot and that the Dehn surgery is of type $(6,31)$. Thus $r=3$, $s=2$, $p=6$, $q=31$ and $|\sigma|=5$.

Recall that by Corollary 5.3 the order of $H_1(\mathcal{M})$ is $30\alpha_0+15\alpha_1+10\alpha_2+6\alpha_3$. But from Fact 1 we have that the order of $H_1(\mathcal{M})$ is $|q|$. Thus we want to find any solutions to

$$30\alpha_0+15\alpha_1+10\alpha_2+6\alpha_3=31.$$

Since $s=2$ we know that $\alpha_1=1$. Using this and dividing both sides by 2 gives

$$15\alpha_0+5\alpha_2+3\alpha_3=8.$$

Since $r=3$ and $|\sigma|=5$ we know $\alpha_2\in\{1,2\}$ and $\alpha_3\in\{1,2,3,4\}$. Clearly then $\alpha_0\le 0$. Suppose $\alpha_0=0$. Then $\alpha_2=1$ and $\alpha_3=1$ are the only solutions. If $\alpha_0<0$ you can check that there are no other solutions.

Equations of this type are called linear Diophantine equations. There is a general algorithm for finding their solutions. This is a fairly standard topic in number theory textbooks [16, 17]. Try some other choices for r, s, p and q.

Exercise 6.1. Let $s=2$, $r=3$, $p=5$, and $q=2$. Show that $\mathcal{M}=S^2(-1,1/2,1/3,5/28)$.

Case 2. Suppose $\sigma=\pm1$. Recall $h=\begin{bmatrix} a & p \\ b & q\end{bmatrix}$. The topological type of the Dehn surgery is determined solely by q and p. We are free to choose a and b so long as $\det h=\pm1$. If we choose $b=rs$ and $a=1$ we get $\det h=pb-aq=q-rsp=\sigma=\pm1$.

From equation $(*)$ we have $h(M_4)\sim pK_1\mp M_3$. We did not need to study $h(L_4)$ in Case 1, but here we do.

$$h(L_4)=rsM_3+L_3\sim rsM_3+K_1-rsM_3\sim K_1.$$

In other words the longitude on V_4 is going to a curve parallel to the knot K.

Now we glue V_4 to V_1. We claim that in this case the result must be a solid torus. Both V_4 and V_1 can be written as $S^1 \times D^2$. For specificity we write

$$V_1 = S_1 \times D_1 \quad \text{and} \quad V_4 = S_4 \times D_4.$$

Let γ be an arc in ∂D_4 and let $A_4 = S_4 \times \gamma$ be the annulus in ∂V_4 that has core L_4 and will be identified to A_1 in ∂V_1. Each copy of the disk D_1 in V_1 meets A_1 in r arcs. We can choose the homeomorphism to take each $* \times \gamma$ arc to a component of $A_1 \cap (*' \times D_1)$. Then the union $V_1 \cup V_4$ can be realized as a product $S^1 \times DD$ where DD is a disk formed by gluing r copies of D_4 to $0 \times D_1$ along copies of γ. See Figure 6.2.

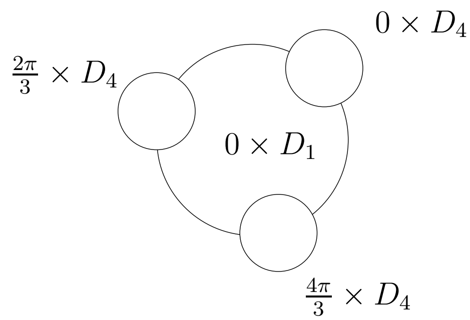

Figure 6.2: Cross section of $V_1 \cup V_4$

Now that we see that $V_1 \cup V_4$ is a solid torus it is immediate that $V_1 \cup V_4 \cup V_2$ is the gluing of two solid tori and hence a lens space. It remains to do some homology calculations to determine which lens space it is.

Remember we have four sets of meridian-longitude pairs. But now we need a fifth since $V_1 \cup V_4$ is a new solid torus. Call these (M_5, L_5). We will compute M_5 in terms of (M_2, L_2), that is we shall solve

$$M_5 = xL_2 + yM_2;$$

we won't need to find L_5. Then we will have the $\mathcal{L}(x, y)$ lens space.

Looking at Figure 6.3 we see that

$$L_1 \sim M_2 + rM_3 \text{ and } M_1 \sim L_2 - sM_3.$$

These homology calculations are still in $V_1 \cup V_2$, the knot complement space. (Figure 6.3 attempts to show V_1 with a small tubular neighborhood of the core drilled out, thus creating a thick torus $(T^2 \times I)$ with a trough dug out along K, which is presented as a cube with the top & bottom sides and the left & right sides respectively identified. Not shown is V_2 which would look like the mirror image. Recall V_1 and V_2 are glued along the annulus A. To the sides of V_1 are diagrams showing how the various meridians and longitudes are related if $r = 3$ and $s = 2$. It will likely take a good while for the reader to see this.)

Now we glue in V_4. Recall that K_1 is an (r, s) curve. Thus,

$$M_4 \sim pK_1 - \sigma M_3 \sim p(rM_1 + sL_1) - \sigma M_3 = prM_1 + psL_1 - \sigma M_3.$$

And finally,

$$\begin{aligned} M_5 &\sim M_1 - \sigma s M_4 \\ &\sim M_1 - \sigma s(prM_1 + psL_1 - \sigma M_3) \\ &\sim (1 - \sigma rsp)M_1 - \sigma ps^2 L_1 + s\sigma^2 M_3 \\ &\sim (1 - \sigma rsp)(L_2 - sM_3) - \sigma ps^2(M_2 + rM_3) + sM_3 \qquad (\sigma^2 = 1) \\ &= (1 - \sigma rsp)L_2 - s(1 - \sigma rsp)M_3 - \sigma ps^2 M_2 - \sigma prs^2 M_3 + sM_3 \\ &= (1 - \sigma rsp)L_2 - \sigma ps^2 M_2 + (-s + \sigma prs^2 - \sigma prs^2 + s)M_3 \\ &= (1 - \sigma rsp)L_2 - \sigma ps^2 M_2, \end{aligned}$$

where these homology calculations are in the new manifold $\mathcal{M}$. If $\sigma = 1$ then $M_5 \sim -qL_2 - ps^2 M_2$; if $\sigma = -1$ then $M_5 \sim +qL_2 + ps^2 M_2$. Therefore, $\mathcal{M} \simeq L(|q|, ps^2)$ as claimed.

Case 3. Suppose $\sigma = 0$. Then $q = rsp$. Since q and p can only have 1 as a common divisor, and $p > 0$ by convention, it must be that $p = 1$. Thus by equation $(*)$ $h(M_4) \sim K_1$. That is the meridian M_4 of V_4 is identified with K_1. Another meridian, M_4', of V_4 will then be identified with K_2.

We construct the union $V_1 \cup V_4 \cup V_2$ in stages. Partition V_4 into two solid cylinders C and C' by choosing two disjoint meridianal disks D and D' in V_4 with $\partial D = M_4$ and $\partial D' = M_4'$. The boundary of C minus the interiors of the two disks is an annulus, call it A_3. See Figure 6.4.

We glue C to V_1 by attaching A_3 to A_1. This space has boundary $D \cup A \cup D'$, which must be a 2-sphere, call it S. Likewise $V_2 \cup C'$ is a manifold whose boundary is a 2-sphere, call it S'. Then $\mathcal{M}$ is formed from $V_1 \cup C$ and $V_2 \cup C'$ by identifying their boundary spheres. Thus $\mathcal{M}$ is the connected some of two manifolds. We will show that $V_1 \cup C$ and $V_2 \cup C'$ are lens spaces with an open 3-ball removed.

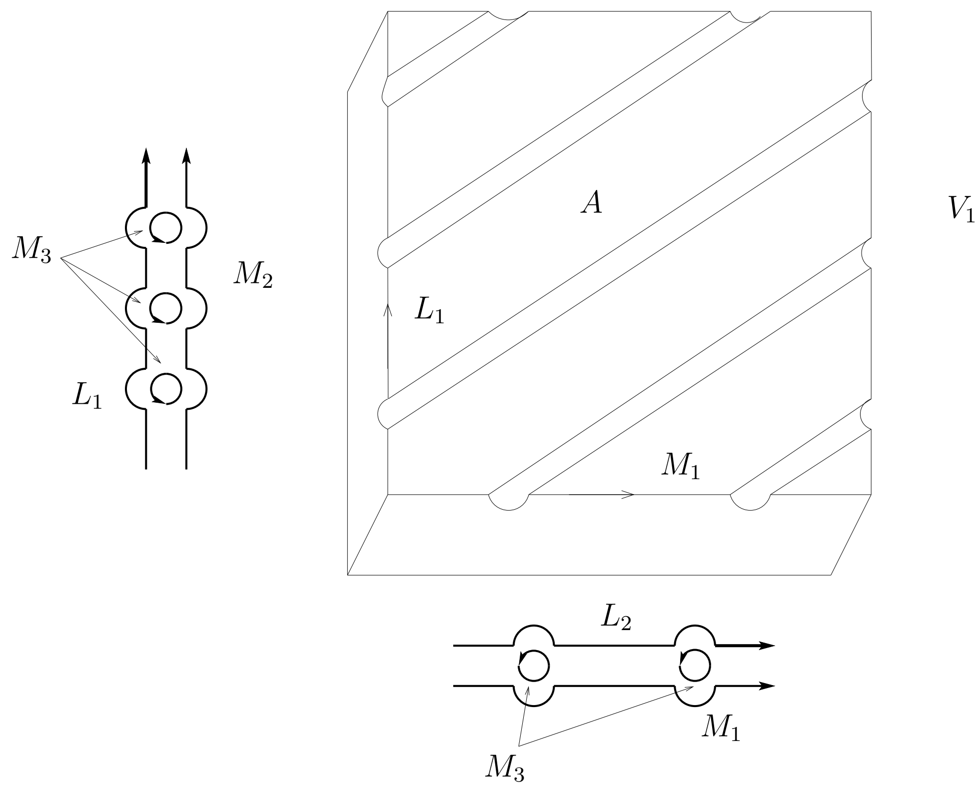

Figure 6.3: V_1 for $r = 3$, $s = 2$

In fact we claim $V_1 \cup C$ is homeomorphic to the lens space $\mathcal{L}(r,s)$ minus an open 3-ball. To see this we glue a 3-ball B to $V_1 \cup C$ and show that this space is $\mathcal{L}(r,s)$. We do this in two steps. Partition B into a solid torus V_B and a solid cylinder C_B as shown in Figure 6.5. Let D_B and D'_B be the disks composing $\partial B \cap C_B$. Let $A_B = \partial B - \mathrm{int}(D_B \cup D'_B)$. Attach C to C_B by identifying D_B to D_1 and D'_B to D_2. Then $C \cup C_B$ is a solid torus. Now attach V_1 to V_B by identifying the annulus A_1 with A_B. As before this forms a solid torus. Thus,

$$V_1 \cup C \cup B = (V_1 \cup V_B) \cup (C \cup C_B)$$

is the union of two solid tori and is a lens space. Since a meridian ∂D_1 of $C \cup C_B$ is identified to an (r,s) curve on $\partial(V_1 \cup V_B)$ the lens space is $\mathcal{L}(r,s)$.

If we attach C' to V_2 we can show that this is homeomorphic to the lens space $\mathcal{L}(s,r)$ minus an open ball. Thus $\mathcal{M}$ is formed by taking the connected

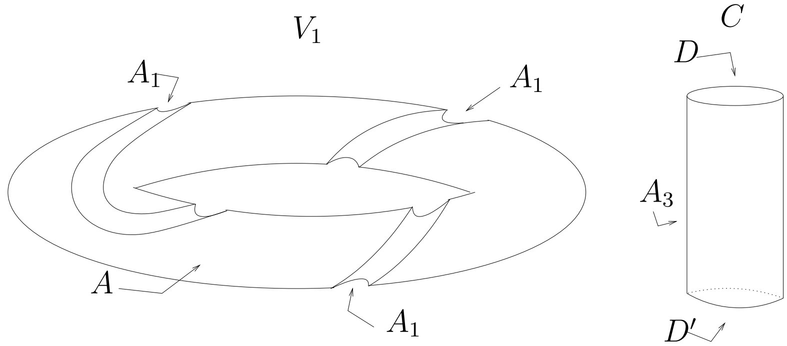

Figure 6.4: V_1 and C

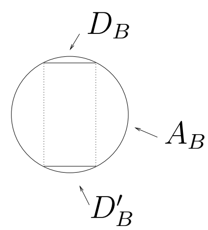

Figure 6.5: The 3-ball B partitioned

sum of $\mathcal{L}(r,s)$ and $\mathcal{L}(s,r)$. Note: It is known that $\mathcal{L}(r,s)\#\mathcal{L}(s,r)$ cannot be given a Seifert fibration. $\square$

This concludes the proof. The figure-8 knot has just four crossings and is not a torus knot. Surgery along the figure-8 knot is the next logical topic to pursue. This turns out to be much more involved than surgery along torus knots. See [21] as a place to start. There is a large and growing literature on this topic.

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