Lorenz like Smale flows on three-manifolds

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A B S T R A C T

In this paper, we discuss how to realize Lorenz like Smale flows (LLSF) on 3-manifolds. It is an extension of M. Sullivan's work about Lorenz Smale flows on S^3. We focus on two questions: (1) Classify the topological conjugate classes of LLSF which can be realized on S^3; (2) Which 3-manifolds admit LLSF? If some 3-manifold admits LLSF, how does it admit LLSF? This paper is in some sense parallel to the work of J. Morgan and M. Wada on Morse–Smale flows on 3-manifolds.

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1. Introduction

Many papers about flows on three-manifolds are about the realization of some types of flows on three-manifolds. J. Morgan [13] studied which three-manifolds admit non-singular Morse–Smale flows (NMSF) using round handle decomposition. Following J. Morgan's work, M. Wada [17] systematically studied how to realize NMSF on S^3. M. Handel and W.P. Thurston [12], Goodman [11] constructed some three-manifolds which admit Anosov flows.

J. Franks [5] systematically studied how to realize non-singular Smale flows (NSF) on S^3 via Lyapunov graph. However, this method doesn't include embedding information. On the other hand, J. Franks [6,7] used homology to describe some embedding information.

Similar to M. Wada's work about the realization of NMSF on S^3, M. Sullivan [16] studied a special type of NSF on S^3. This type of NSF is called Lorenz Smale flows. A Lorenz Smale flow is a Smale flow with three basic sets: a repelling orbit, an attracting orbit, and a non-trivial saddle set modeled by a Lorenz template.

In this paper, we discuss how to realize Lorenz like Smale flows (LLSF) on 3-manifolds. It is an extension of M. Sullivan's work [16] about Lorenz Smale flows on S^3. A LLSF is a NSF with three basic sets: a repelling orbit r, an attracting orbit a, and a non-trivial saddle set modeled by a Lorenz like template. Disregarding embedding information, there are three types of Lorenz like templates: Lorenz template $\mathcal{L}(0,0)$, horseshoe template $\mathcal{L}(0,1)$, and template $\mathcal{L}(1,1)$. We denote by $\mathcal{L}(i,j)$ a Lorenz like template whose saddle set is modeled by Lorenz like template $\mathcal{L}(i,j)$. We focus on two questions: (1) Classify the topological conjugate classes of LLSF which can be realized on S^3; (2) Which 3-manifolds admit LLSF? If it admits LLSF, how does it admit LLSF? B. Campos, A. Cordero, J. Martínez Alfaro, P. Vindel [4] studied the same problem as question (2)

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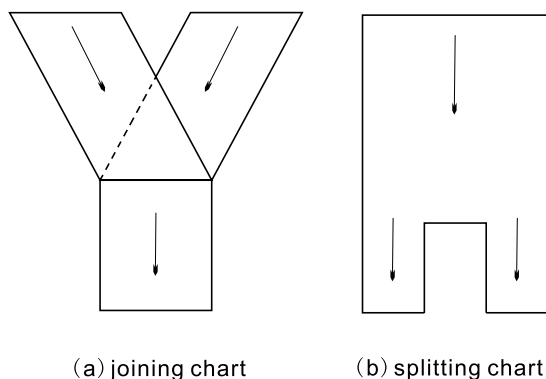
(a) joining chart (b) splitting chart

Fig. 1.

but where the saddle set is a single periodic orbit. The method we use is similar to that of M. Sullivan. The difference between this paper and M. Sullivan's paper is that we try to use only combinatorial tools in the discussion. We hope that these discussions will shed some lights to a systematically study about NSF on three-manifolds.

The main theorems are the following.

Theorem 1. *For an* $\mathcal{L}(0, 1)$ *Lorenz like Smale flow on* S^3 *the following and only the following configurations are realizable. The link* $a \sqcup r$ *is either a Hopf link or a trefoil and meridian. In the later case the saddle set is modeled by a standard embedded* $\mathcal{L}(0, 1)$ *Lorenz like template, i.e. the saddle set is modeled by embedded* $\mathcal{L}(0, 1)$ *and the cores of both bands are unknotted and unlinked each other. In the former case, there are three possibilities:* (1) *The saddle set is standardly embedded.* (2) *The saddle set is modeled by embedded* $\mathcal{L}(2p + 2q - 2, 2p + 2q - 1)$. *The cores of two bands are two parallel* (p, q) *torus knots. p, q are any coprime integers.* (3) *The saddle set is modeled by embedded* $\mathcal{L}(0, 2p + 2q - 1)$. *The core of the twisted band is a* (p, q) *torus knot, the core of the other band is unknotted and unlinked with the former one. p, q are any coprime integers.*

Theorem 2. *For an* $\mathcal{L}(1, 1)$ *Lorenz like Smale flow on* S^3 *the following and only the following configurations are realizable. The link* $a \sqcup r$ *is a link which is composed of a trivial knot and a* $(p, 3)$ *torus knot in the boundary of a solid torus neighborhood of the trivial knot. Here p is any integer such that p, 3 are coprime. The saddle set is modeled by embedded* $\mathcal{L}(2n + 1, 4n + 1)$ *for any n. The linking number of the cores of these two bands is* $2n$, *the core of one band is unknotted and the core of the other band is a* $(2, 2n + 1)$ *torus knot.*

Theorem 3. *The closed orientable 3-manifolds which admit* $\mathcal{L}(0, 1)$ *Lorenz like Smale flows are the following and only the following:* S^3, $S^2 \times S^1$, *lens space* $L(p, q)$, *Seifert manifolds* $S^2(\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha})$, $S^2(\frac{1}{2}, \frac{1}{3}, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2})$, *and* $L(3, 1)\sharp L(2, 1)$. *Here* α, β *are any integers such that* α *and* β *are coprime,* $\alpha \neq 0$ *and* α_i, β_i $(i = 1, 2)$ *also satisfies this restriction.*

Theorem 4. *The closed orientable 3-manifolds which admit* $\mathcal{L}(1, 1)$ *Lorenz like Smale flows are the following and only the following:* S^3; *lens space* $L(3, 1)$; $L(3, 1)\sharp Y$, *here Y is* $S^2 \times S^1$ *or any lens space* $L(p, q)$; *Seifert manifolds* $S^2(\frac{1}{3}, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2})$. *Here* α_i, β_i $(i = 1, 2)$ *are any integers such that* α_i *and* β_i *are coprime,* $\alpha_i \neq 0$.

The descriptions of LLSF on the 3-manifolds in Theorems 3, 4 can be found in Section 4.

2. Preliminaries

The most popular template is Lorenz template which was used by R.F. Williams [18] as a model to analyze Lorenz attractor. Template theory was first introduced to dynamics by R.F. Williams and J. Birman in their papers [1,2].

Definition 2.1. *A* template (T, ϕ) *is a smooth branched 2-manifold* T, *constructed from two types of charts, called* joining charts *and* splitting charts, *together with a semi-flow. A semi-flow is the same as flow except that one cannot back up uniquely. In Fig. 1 the semi-flows are indicated by arrows on charts. The gluing maps between charts must respect the semi-flow and act linearly on the edges.*

As [16], by a *simple Smale flow*, we mean a non-singular Smale flow with three basic sets: a repelling orbit, an attracting orbit, and a non-trivial saddle set. If the non-trivial saddle set can be modeled by *Lorenz like template*, we call the simple Smale flow a *Lorenz like Smale flow*.
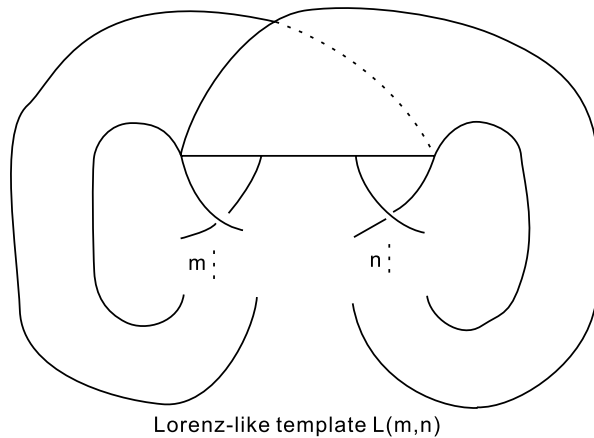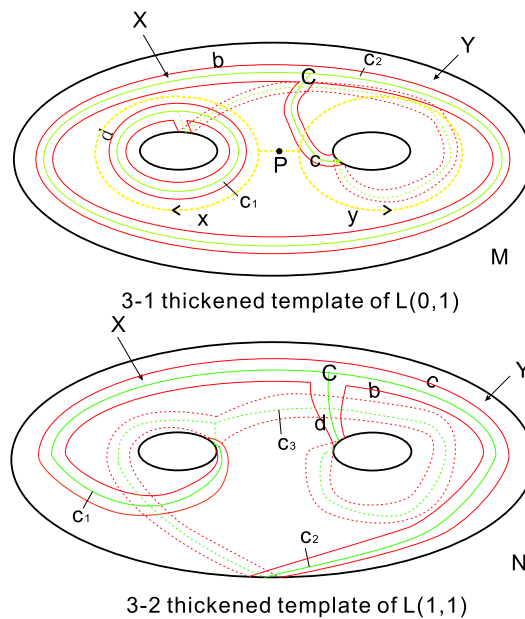
Lorenz-like template L(m,n)

Fig. 2.

3-1 thickened template of L(0,1)

3-2 thickened template of L(1,1)

Fig. 3.

Definition 2.2. For $m, n \in Z$, denote by $\mathcal{L}(m, n)$ the (m, n) type *Lorenz like template*. Here m, n are half-twists number respectively, as Fig. 2 shows.

As far as topological classification is concerned, there are only three Lorenz like templates: $\mathcal{L}(0, 0)$, $\mathcal{L}(0, 1)$ and $\mathcal{L}(1, 1)$. $\mathcal{L}(0, 0)$ is *the Lorenz template*; $\mathcal{L}(0, 1)$ is *the Smale horseshoe template*. M. Sullivan studied simple Smale flow on S^3 in the case the saddle set can be modeled by $\mathcal{L}(0, 0)$. So we can restrict our discussion to $\mathcal{L}(0, 1)$ and $\mathcal{L}(1, 1)$. If we embed these templates to S^3, then we can denote them by $\mathcal{L}(m, n)$. Here m, n are half-twists number of two bands of the template.

For a Lorenz like Smale flow, an isolating neighborhood of the saddle set can be regarded as a thickened Lorenz like template. The isolating neighborhood is a genus two handlebody with flow. The points in the boundary of the isolating neighborhood where the flow is transverse outward (inward) is called the *exit set* (*entrance set*). The exit set and the entrance set intersect in a finite union of closed curves. There are two cases: $\mathcal{L}(0, 1)$ and $\mathcal{L}(1, 1)$, we denote their neighborhoods by M and N respectively. The exit set and the entrance set of M (N) are denoted by X and Y respectively. The core of X is denoted by C; the intersection of X and Y, which consists of closed curves b, c and d, is denoted by S. The fundamental group of M (N) is $Z * Z = \langle x, y \rangle$. The basic point is P. See Fig. 3.

3. The realization of Lorenz like Smale flows on S^3

The basic facts about three-manifolds used in this paper can be found in [9,10].
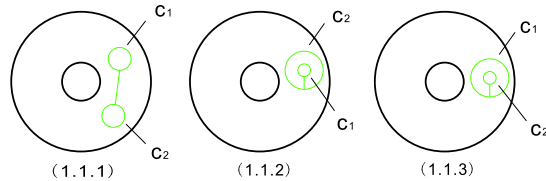
Fig. 4.

Lemma 3.1. *Let $W = D^2(\frac{q_1}{p_1}, \frac{q_2}{p_2})$ be a Seifert manifold with orbit-manifold a disk and with two exceptional fibers. $\frac{q_1}{p_1}$, $\frac{q_2}{p_2}$ are slopes of these two exceptional fibers. p_1 and p_2 are coprime. Given two integers s, t such that $p_1 s - p_2 t = 1$. Then W is a knot complement space if and only if $q_1 \equiv -t \pmod{p_1}$, $q_2 \equiv s \pmod{p_2}$ or $q_1 \equiv t \pmod{p_1}$, $q_2 \equiv -s \pmod{p_2}$. Furthermore, if W is a knot complement space, the knot must be a (p_1, p_2) type torus knot.*

Proof. $\pi_1(W) = \{x, y \mid x^{p_1} = y^{p_2}\}$. Hence if W is a knot complement space the knot must be (p_1, p_2) torus knot. See [3, Lemma 15.37, Corollary 15.23]. Some standard computations show a (p_1, p_2) type knot complement space can be written as $D^2(\frac{-t}{p_1}, \frac{s}{p_2})$. By Proposition 2.1, Theorem 2.3 in Chapter 2 of [8] and computations above, it is easy to show W is a knot complement space if and only if $q_1 \equiv -t \pmod{p_1}$, $q_2 \equiv s \pmod{p_2}$ or $q_1 \equiv t \pmod{p_1}$, $q_2 \equiv -s \pmod{p_2}$. In these cases, the knot must be a (p_1, p_2) type torus knot. $\quad\square$

Let A (R) be a canonical neighborhood of a closed attractor (repeller) and a (r) is the core of A (R). A (R) is a solid torus with flow which is transverse inward (outward) to ∂A (∂R). To construct Lorenz like Smale flows on S^3, we first attach the closure of the exit set of a thickened Lorenz like template M (N) to ∂A. We denote the attached space MA (NA).

Lemma 3.2. *MA (NA) can be regarded as part of a Lorenz like Smale flow on S^3 if and only if the interior of MA (NA) is homeomorphic to the complement of a knot in S^3.*

Proof. If MA (NA) is a building block of a Lorenz like Smale flow on S^3, then MA can be regarded as the closure of the complement of R in S^3. The core of R is a knot in S^3. The "only if" part is proved.

The "if" part is easy to prove by attaching R to MA (NA) such that its core is the knot and its boundary is attached to the boundary of MA (NA). $\quad\square$

Lemma 3.2 is the key. We will discuss all possible configurations of MA and NA using Lemma 3.2 to decide which ones can be realized in a Lorenz like Smale flow on S^3.

Attaching i-handle is a useful surgery in our arguments below. An i-handle, $1 \leqslant i \leqslant 3$, is an i-ball of form $D^i \times D^{3-i}$. The index i indicates the intention to attach it to something else along the $S^{i-1} \times D^{3-i}$ part of its boundary. More detail can be found in Chapter 11 of [15].

Case 1. *MA (the proof of Theorem 1).*

In this case, the core C is two circles c_1, c_2 connected by an arc l, c_1 is homotopic to x in M. The exit set X is the union of three sets: $\overline{c_1}$, $\overline{c_2}$ and $\overline{l}$. Here $\overline{c_1}$, $\overline{c_2}$ and $\overline{l}$ are homeomorphic to $c_1 \times [0, 1]$, $c_2 \times [0, 1]$ and $l \times [0, 1]$ respectively.

Case 1.1. c_1, c_2 both are inessential in ∂A. There are three subcases, as Fig. 4 shows.

We attach 2-handles $\overline{D_1}$ and $\overline{D_2}$ to M along $\overline{c_1}$ and $\overline{c_2}$ respectively, we call it M'. In subcase (1.1.1), MA is homeomorphic to the manifold which is constructed by attaching 1-handle $\overline{D_3}$ to M' in $\overline{l}$; in subcase (1.1.2), MA is homeomorphic to the manifold which is constructed by attaching 1-handle $\overline{D_3}$ to M' in $\overline{D_1}$; in subcase (1.1.3), MA is homeomorphic to the manifold which is constructed by attaching 1-handle $\overline{D_3}$ to M' in $\overline{D_2}$. Actually, in any subcase above, MA is homeomorphic to solid torus with core a. So in Case 1, any subcase can be realized as Lorenz like Smale flow on S^3 and $a \sqcup r$ is a Hopf link in S^3. However, as flows, any one of the three subcases is not topologically conjugate to the others. In this case, the saddle set is standardly embedded, i.e. the saddle set is modeled by embedded $\mathcal{L}(0, 1)$ and the cores of both bands are unknotted and unlinked to each other.

Case 1.2. c_2 is inessential in ∂A, c_1 is essential in ∂A, as Fig. 5(1.2) shows.

Here c_1 is a (p, q) type curve in ∂A. Because c_2 is inessential, to visualize MA, we can attach a 2-handle to M along $\overline{c_2}$, we call it M'. M' is a solid torus, c_1 is a longitude in $\partial M'$. Then we can see that MA is homeomorphic to a solid torus, so this case can be realized as Lorenz like Smale flow on S^3 and $a \sqcup r$ is a Hopf link in S^3, c_1 is a (p, q) type torus knot. In this case, the saddle set is modeled by embedded $\mathcal{L}(2p + 2q - 2, 2p + 2q - 1)$. The cores of two bands are two parallel (p, q) torus knots.

Case 1.3. c_1 is inessential in ∂A, c_2 is essential in ∂A, as Fig. 5(1.3) shows. Here c_2 is a (p, q) type curve in ∂A. By the same analysis in Case 1.2, we can show that this case can be realized as Lorenz like Smale flow on S^3 and $a \sqcup r$ is a Hopf

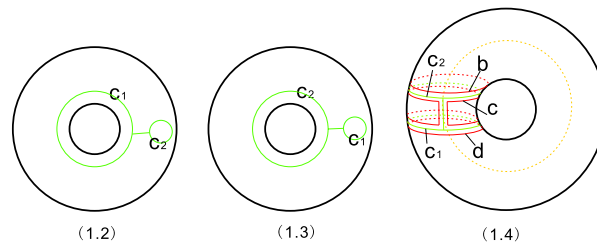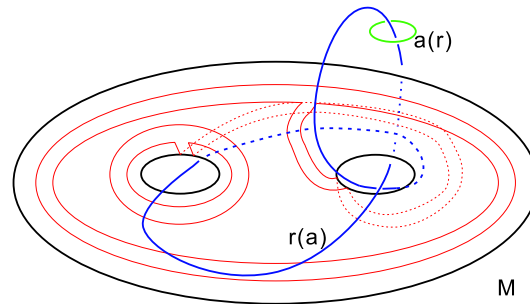**Fig. 5.**

Fig. 6.

link in S^3, c_2 is a (p, q) type torus knot. In this case, the saddle set is modeled by embedded $\mathcal{L}(0, 2p + 2q - 1)$. The core of one band is a (p, q) torus knot, the core of the other band is unknotted and unlinked with the former one.

Case 1.4. c_1, c_2 both are essential in ∂A, as Fig. 5(1.4) shows. Here c_1 and c_2 are two parallel (p, q) type curves in ∂A.

It's not difficult to see that c bounds a disk D in ∂A from Fig. 5(1.4). We denote a neighborhood of the disk by $B = D \times I$, $I = [0, 1]$ and denote by MB the manifold $M \cup B$ via attaching B to M along $\partial D \times I$ and $\bar{c}$. Here $\bar{c}$ is an annulus neighborhood of c in ∂M. Hence $\overline{MB^c} \cong A \cup_\varphi R$, ∂MB is a torus and is divided to two annuluses T_1, T_2 by d, b. Here $\varphi : A_1 \to A_2$ is a homeomorphism, A_1 is an annulus in ∂A and A_2 is an annulus in ∂R.

Using Van Kampen's theorem, we can get the fundamental group of MB,

$$\pi_1(MB) = \left\{ x, y \mid x^{-1}yyx^{-1}y^{-1} = 1 \right\}.$$

Here x, y are the generators of $\pi_1(M)$ shown in Fig. 3. It is easy to know

$$\pi_1(MB) = \left\{ x, y \mid y^3 = (xy)^2 \right\}.$$

It is a trefoil knot group. It can be observed in Fig. 3-1 that we have $MB \hookrightarrow S^3$ with a torus as its boundary. By [3, Lemma 15.37, Corollary 15.23], $MB \cong K^c \subset S^3$, here K is a trefoil knot. ∂MB is a torus, by the solid torus theorem [15, p. 107], $\overline{MB^c} \cong A \cup_\varphi R$ is a solid torus.

We attach a 2-handle to MB along d. Denote it by MBD. By Fig. 3, we know that $\pi_1(MBD) = \{x, y \mid x^{-1}yyx^{-1}y^{-1} = 1, x = 1\}$ is a trivial group. Since $\partial MBD \cong S^2$, MBD is a three ball. By Dehn surgery theorem, if we attach a solid torus to MB along ∂MB such that the new manifold is S^3 and the core of the solid torus is a trefoil knot in S^3, then d is a meridian of the boundary of the solid torus.

We denote the core of A_i by a_i, a_i is a (p_i, q_i) type curve in ∂A (∂R), for $i = 1, 2$. Then $\pi_1(A \cup_\varphi R) = \{t, s \mid t^{p_1} = s^{p_2}\}$, here $\pi_1(A) = \{t\}$, $\pi_1(R) = \{s\}$. On the other hand, since $A \cup_\varphi R$ is a solid torus, $\pi_1(A \cup_\varphi R) \cong Z$. So we get $p_1 = 1$ or $p_2 = 1$. If $p_1 = 1$, we can assume $q_1 = 0$. $A \cup_\varphi R$ is a solid torus with the same core as R. The entrance and exit sets are separated by two parallel (p_2, q_2) type curves in the boundary of this solid torus. In order that $A \cup_\varphi R$ with flow could be attached to MB to form flow on S^3, (p_2, q_2) must be $(0, 1)$. In this case, the repeller r is a trefoil knot in S^3, the attractor a is a meridian of r. Similarly, if $p_2 = 1$, the attractor a is a trefoil knot in S^3, the repeller r is a meridian of a, and the saddle set is modeled by a standard embedded $\mathcal{L}(0, 1)$ Lorenz like template. Fig. 6 shows how the core of the attractor, the repeller, and the thickened template M are seen in S^3.

Case 2. *NA* (the proof of Theorem 2).

In this case, the core C is a graph with two vertexes and three arcs c_1, c_2, c_3, as Fig. 3-2 shows. Here c_1 and c_3 are symmetric, i.e., there is a homeomorphism $F : N \to N$ such that $F(c_1) = c_3$, $F(c_3) = c_1$ and F preserves the flow lines of N.

Case 2.1. All closed curves in C are inessential in ∂A. Since c_1 and c_3 are symmetric, there are two subcases, see Fig. 7.

Let $\pi_1(A) = \{t\}$. In any subcase above, $\pi_1(NA) = \{x, y, t \mid xy^{-1}x = 1, \ xy = 1\} \cong Z_3$. So in Case 2.1, the interior of NA cannot be a knot complement in S^3. Thus we cannot obtain a Lorenz like Smale flow on S^3 in this case.
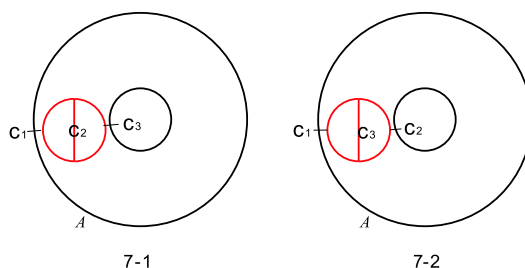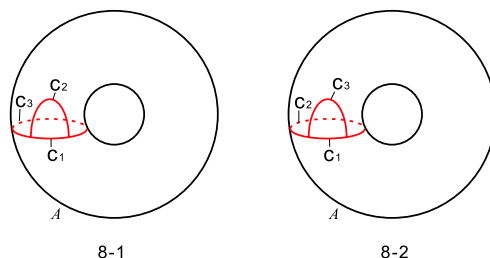
7-1   7-2

Fig. 7.

8-1   8-2

Fig. 8.

Case 2.2. One closed curve in C is inessential in ∂A and the other two curves are essential in ∂A. There are two subcases, see Fig. 8. In the subcase 8-1, $c_1 c_3$ and $c_2 c_3$ are two parallel (p, q) curves in ∂A; in the subcase 8-2, $c_1 c_2$ and $c_3 c_2$ are two parallel (p, q) curves in ∂A. Let $\pi_1(A) = \{t\}$.

Subcase 2.2.1. As Fig. 8-1 shows, we have that c bounds a disk D in ∂A. We denote a neighborhood of the disk by $B = D \times I$, $I = [0, 1]$ and denote by NB the manifold $N \cup B$ through attaching B to N along $\partial D \times I$ and $\bar{c}$. Hence $\overline{NB^c} \cong A \cup_\varphi R$, ∂NB is a torus and is divided to two annuluses T_1, T_2 by d, b. Here $\varphi : A_1 \to A_2$, A_1 is an annulus in ∂A and A_2 is an annulus in ∂R.

$\pi_1(NB) = \{x, y \mid xy^{-1}x = 1\} = \{x\} \cong Z$. Hence NB is a solid torus, so we can take d as $(3, 1)$ type simple closed curve in ∂NB. d is homotopic to x^3 in the solid torus. NA can be obtained by attaching A to NB along T_1. Let s and t be two integers such that $ps - qt = 1$. Thus $NA \cong D^2(\frac{-2}{3}, \frac{-t}{p})$. By Lemma 3.1, if NA is homeomorphic to a knot complement space, the knot must be a $(3, p)$ type torus knot and 3, p are coprime. In this case $S^3 = NA \cup R$, d is parallel to r. Hence $NB \cup R$ is homeomorphic to a solid torus as the same core as NB. Then $(NB \cup R) \cup A$ is the standard genus one Heegaard splitting of S^3. Choosing a suitable coordinate of R, we have $q = 3$ or $q = -3$.

If $p = 3k - 1$, then $3 \times k - p \times 1 = 1$. By Lemma 3.1, NA is homeomorphic to a knot complement space if and only if $t \equiv k(p)$. Similarly, if $p = 3k + 1$, NA is homeomorphic to a knot complement space if and only if $t \equiv k(p)$. a is a $(3, 1)$ type simple closed curve in ∂NB and is (p, q) type simple closed curve in ∂A, hence NA is homeomorphic to a knot complement space if and only if $p \mid (qk + 1)$. By Lemma 3.2 and arguments above, if and only if $p = 3k - 1$, $q = -3$ or $p = 3k + 1$, $q = 3$, NA can be regarded as part of a Lorenz like Smale flows on S^3.

So in this subcase, if it can be realized as Lorenz like Smale flow on S^3, then $a \sqcup r$ is a link which is composed of a trivial knot a and a $(p, 3)$ torus knot r in the boundary of a standard solid torus neighborhood of the trivial knot a. In this case, the saddle set is modeled by embedded $\mathcal{L}(2n + 1, 4n + 1)$ for any n. The linking number of these two bands is $2n$, the core of one band is unknotted and the core of the other band is a $(2, 2n + 1)$ torus knot.

Subcase 2.2.2. As Fig. 8-2 shows, by similar discussion as in Subcase 2.2.1, if it can be realized as Lorenz like Smale flow on S^3, then $a \sqcup r$ is a link which is composed of a trivial knot r and a $(p, 3)$ torus knot a in the boundary of a standard solid torus neighborhood of the trivial knot r.

Case 2.3. All three simple closed curves in C are essential in ∂A. Some easy arguments about graphs in T^2 tell us that this case doesn't exit.

4. 3-manifolds which admit Lorenz like Smale flows

In this section we study when a closed orientable 3-manifold W admits Lorenz like Smale flows. This question is equivalent to how to combine M (N), A and R together, i.e. the entrance sets must be attached to the exit sets. To combine M (N), A and R together, we first consider all possible MA (NA), then obtain 3-manifold W by attaching R to MA (NA). Hence we can organize our discussion just like Section 3.
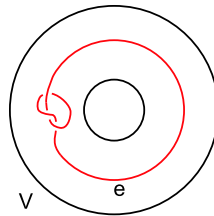
Case 1. *MA* (the proof of Theorem 3).

Fig. 9.

Case 1.1. c_1, c_2 both are inessential in ∂A. MA is homeomorphic to a solid torus with core a. So all lens spaces $L(p,q)$ and $S^2 \times S^1$ admit Lorenz like Smale flows. Let $U \cup V$ is a genus one Heegaard splitting of $L(p,q)$ or $S^2 \times S^1$. Then a can be regarded as the core of U; r can be regarded as the core of V.

Cases 1.2, 1.3. One of c_1, c_2 is inessential in ∂A, the other isn't. In these two cases, we also obtain all lens spaces $L(p,q)$ and $S^2 \times S^1$ admit Lorenz like Smale flows. Let $U \cup V$ be a genus one Heegaard splitting of $L(p,q)$ or $S^2 \times S^1$. Then a can be regarded as the core of U; r can be regarded as the core of V.

Case 1.4. c_1, c_2 both are essential in ∂A. In this case, all lens spaces $L(p,q)$ and all 3-manifolds obtained by doing Dehn filling from trefoil knot complement space admit Lorenz like Smale flows.

For lens space $L(p,q)$, let $U \cup V$ be a genus one Heegaard splitting. Then a (r) can be regarded as the core of U; r (a) can be regarded as the closed curve e in V. See Fig. 9.

Using L. Moser's result [14], it is easy to shows that the 3-manifolds obtained by doing Dehn filling from trefoil knot complement space are Seifert manifolds $S^2(\frac{1}{3}, \frac{1}{2}, \frac{\beta}{\alpha})$ and $L(3,1) \sharp L(2,1)$. Here α, β are any integers such that α and β are coprime, $\alpha \neq 0$.

For Seifert manifold $S^2(\frac{1}{3}, \frac{1}{2}, \frac{\beta}{\alpha})$, a (r) is the (α, β) type singular fiber and r (a) is a $(0,1)$ simple closed curve. The chart is given by regular fiber in ∂A (∂R), the chart of the regular fiber is $(1,0)$.

For $L(3,1) \sharp L(2,1)$, the space can be taken as $U = \overline{D - D_1 \sqcup D_2} \times S^1$ attached with three solid torus N, N_1, N_2. Here D is a disk, D_1 and D_2 are two disjoint disks in the interior of D. $\partial D \times S^1$ and $\partial D_i \times S^1$ admit natural multiple charts. N is attached to U along $\partial D \times S^1$, i.e. the meridian of N attaches to $(0,1)$ simple closed curve; N_i is attached to U along $\partial D_i \times S^1$, i.e. the meridian of N_1 is attached to $(3,1)$ simple closed curve and the meridian of N_2 is attached to $(2,1)$ simple closed curve. Then $a \sqcup r$ ($r \sqcup a$) is isometric to $C(N) \sqcup pt \times S^1$. Here pt is any given point in $\overline{D - D_1 \sqcup D_2}$ and $C(N)$ is the core of N.

$S^2(\frac{1}{3}, \frac{1}{2}, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2})$ also admits this type LLSF. a is the (α_1, β_1) type of singular fiber and r is the (α_2, β_2) type of singular fiber. Here α_i, β_i ($i = 1, 2$) are any integers such that α_i and β_i are coprime, $\alpha_i \neq 0$.

Case 2. *NA* (the proof of Theorem 4).

Case 2.1. All simple closed curves in C are inessential in ∂A. MA is homeomorphic to the connected sum of a solid torus U and $L(3,1)$. So all $L(3,1) \sharp Y$ admit Lorenz like Smale flows. Here Y is any lens space $L(p,q)$ or $S^2 \times S^1$. $U \cup R$ is a genus one Heegaard splitting of $L(p,q)$ or $S^2 \times S^1$. Then a can be regarded as the core of U; r is the core of R.

Case 2.2. One closed curve in C is inessential in ∂A and the other two curves are essential in ∂A.

Subcase 2.2.1. c bounds a disk D in ∂A. d and b are two parallel $(3,1)$ type simple closed curves in ∂NB.

If d bounds a disk in A, NA is homeomorphic to the connected sum of a solid torus U and $L(3,1)$. Similar to Case 2.1, all $L(3,1) \sharp Y$ admit Lorenz like Smale flows. Here Y is any lens space $L(p,q)$ or $S^2 \times S^1$ such that p and 3 are coprime. $U \cup R$ is a genus one Heegaard splitting of $L(p,q)$ or $S^2 \times S^1$. Then a can be regarded as the core of U; r is the core of R. It isn't conjugate to Case 2.1.

If d doesn't bound a disk in A, NA is homeomorphic to $D^2(\frac{-2}{3}, \frac{-t}{p})$. Hence Seifert manifold $S^2(\frac{-2}{3}, \frac{-t}{p}, \frac{\beta}{\alpha})$ and $L(3,1) \sharp L(p,q)$ admit Lorenz like Smale flows. Here α, β are any integers such that α and β are coprime, $\alpha \neq 0$.

For Seifert manifold $S^2(\frac{-2}{3}, \frac{-t}{p}, \frac{\beta}{\alpha})$, a is the $(p, -t)$ type singular fiber and r is the (α, β) type singular fiber.

For $L(3,1) \sharp L(p,q)$, let $U \cup V$ be a genus one Heegaard splitting of the lens space $L(p,q)$. Then a can be regarded as the core of U and r can be regarded as the core of V.

Subcase 2.2.2. As Fig. 8-2 shows, it is the same as Subcase 2.2.1 if we exchange the roles of a and r.

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