# Section 7.4
# Basics of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

This course is not heavy on theory, but in this section we lightly cover the basic theory that is needed. The end of Section 7.1 stated theorem (7.1.2) that linear systems of first order differential equations with prescribed initial values, under suitable assumptions, have unique solutions. The proof of this is covered in more advanced courses (MATH 505). If you took MATH 452, we covered the one-dimensional version of this; see also Section 2.8 of the textbook for MATH 305/405. In this this section, we study the form that solution come in, using ideas from linear algebra.

**Theorem 7.4.1.** Suppose $\mathbf{x}_1(t)$, $\mathbf{x}_2(t), \ldots, \mathbf{x}_k(t)$ are $n \times 1$ column vectors of functions that are solutions to $\mathbf{x}' = A\mathbf{x}$. ($A$ is an $n \times n$ matrix of constants or functions.) Then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

is also a solution, where the $c_i$ are real or complex constants.

*Proof.* Just plug in.

$$\begin{aligned} \mathbf{x}' &= c_1\mathbf{x}_1' + c_2\mathbf{x}_2' + \cdots + c_k\mathbf{x}_k' \\ &= c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_kA\mathbf{x}_k \\ &= A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) \\ &= A\mathbf{x} \end{aligned}$$

□

**Theorem 7.4.2.** Suppose $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ are linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$, where $A$ is $n \times n$. Let $\mathbf{y}(t)$ be any other solution. Then there exist unique constants such that $\mathbf{y} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$.

*Outline of Proof.* Pick $t_0 \in \mathbb{R}$ where the solutions are valid. Let $\mathbf{b} = \mathbf{y}(t_0)$. Let $X = [\mathbf{x}_1(t_0) \cdots \mathbf{x}_n(t_0)]$. Then

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = X^{-1}\mathbf{b}$$

By Theorem 7.1.2 we are done. □

**Definition.** A set of $n$ linearly independent solutions, $\{\mathbf{x}_1(t), \dots \mathbf{x}_n(t)\}$ is called a **fundamental solution set**. It is a basis for the **solution space**. The matrix $X(t) = [\mathbf{x}_1(t) \cdots \mathbf{x}_n(t)]$ is called a **fundamental matrix**. When working with an initial value $t_0$, if we have that $X(t_0)$ is the identity matrix, then $X(t)$ is called the **special fundamental matrix**. (See Section 7.7.)

**On the Wronskian.** Next we need to deal with linear independence. The right tool for this is the Wronskian. A version of this was covered in MATH 305. I'll review this first with a couple of examples.

**Example (From Section 3.2).** Show $e^t$ and $e^{2t}$ are linearly independent, that one is not a multiple of they other. To do this we computer their Wronskian $W(e^t, e^{2t})$ as follows.

$$W(e^t, e^{2t}) = \det \begin{bmatrix} e^t & e^{2t} \\ (e^t)' & (e^{2t})' \end{bmatrix} = 2e^{3t} - e^{3t} = e^{3t} \neq 0.$$

Hence, $e^t$ and $e^{2t}$ are linearly independent.

**Example (From Section 3.2).** Show that $\sin t$ and $\cos t$ are linearly independent.

$$W(\sin t, \cos t) = \det \begin{bmatrix} \sin t & \cos t \\ (\sin t)' & (\cos t)' \end{bmatrix} = -\sin^2 t - \cos^2 t = -1 \neq 0.$$

Hence, $\sin t$ and $\cos t$ are linearly independent.

The Wronskian of three functions is defined like this

$$W(f, g, h) = \det \begin{bmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{bmatrix}.$$

The three functions are linearly independent if and only if the Wronskian is not zero. See Section 4.1 for more on this.

In Chapter 7 we are dealing with vector valued functions. The Wronskian plays the same role, but takes a slightly different form. Given $n$ column vectors, each $n \times 1$, use them to form an $n \times n$ matrix. The Wronskian is just the determinant of this matrix. Here is an example.

**Example.** In the lecture for 7.1 we solved

$$x' = 2x - 3y$$

$$y' = x - 2y$$

We got

$$x(t) = 3C_1e^t + C_2e^{-t}$$

$$y(t) = C_1e^t + C^2e^{-t}$$

Suppose, $x(0) = 1$ and $y(0) = 0$. Then

$$2C_1 + C_2 = 1$$

and

$$C_1 + C_2 = 0.$$

Thus, $C_2 = -C_1$ and hence $2C_1 = 1$. Thus $C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{2}$.

Now suppose $x(0) = 0$ and $y(0) = 1$. Then we get $C_1 = -\frac{1}{2}$ and $C_2 = \frac{3}{2}$. Now we have a pair of solutions.

$$\mathbf{v}_1(t) = \begin{bmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} \\ \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} \end{bmatrix} \ \& \ \mathbf{v}_2(t) = \begin{bmatrix} -\frac{3}{2}e^t + \frac{3}{2}e^{-t} \\ \\ -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}$$

We can now compute their Wronskian to check that they are linearly independent.

$$W(\mathbf{v}_1, \mathbf{v}_2) = \det \begin{bmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} & -\frac{3}{2}e^t + \frac{3}{2}e^{-t} \\ \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} & -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix} = 1 \neq 0.$$

Now, for an arbitrary collection of $n$ $n \times 1$ column vectors of functions, it can happen that the Wronskian is sometimes zero and sometimes not zero. Notice this never happened in our examples.

**Theorem 7.4.3.** If $\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)$ are solutions to an $n \times n$ system $\mathbf{x}' = A\mathbf{x}$ on some interval $(a, b)$, then for $t \in (a, b)$ the Wronskian $W(\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t))$ is always zero or never zero. For the proof see Problems 2, 8 and 9 in this section. (Could be a starting for a project.)

Back to our example. Let $t = 0$. Then $W(\mathbf{v}_1(0), \mathbf{v}_2(0)) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$. Done.

**Complex Case.** When dealing with systems that involve complex numbers (as eigenvalues) the following theorem will be useful.

**Theorem 7.4.5.** If $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $A$, $\mathbf{u}$ and $\mathbf{v}$ have real entries, then $\mathbf{u}' = A\mathbf{u}$ and $\mathbf{v}' = A\mathbf{v}$.

*Proof.*

$$\begin{aligned}\mathbf{x} - A\mathbf{x} = \mathbf{0} \quad &\Longrightarrow \quad (\mathbf{u} + i\mathbf{v})' - A(\mathbf{u} + i\mathbf{v}) = \mathbf{0} \\ &\Longrightarrow \quad (\mathbf{u}' - A\mathbf{u}) + i(\mathbf{v}' - A\mathbf{v}) = 0 \\ &\Longrightarrow \quad \mathbf{u}' = A\mathbf{u} \& \mathbf{v}' = A\mathbf{v}\end{aligned}$$

□

**Note.** In Section 7.9 we will look at the nonhomogeneous situation, $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.