Section 7.5
Solutions to $\mathbf{x}' = A\mathbf{x}$, Real Distinct Eigenvalues

Assume A is an $n \times n$ matrix of real numbers that has n real, distinct eigenvalues. We will show how to solve $\mathbf{x}' = A\mathbf{x}$. Recall that equation like $y' = ay$ had solutions of the form $y(y) = Ce^{at}$. This leads us to guess that for $\mathbf{x}' = A\mathbf{x}$ we might try solutions of the form $\mathbf{x} = \mathbf{v}e^{rt}$, where $\mathbf{v}$ is a column vector of constants. This is also the form of solutions we found in §7.1 looking at 2×2 systems.

We plug this into $\mathbf{x}' = A\mathbf{x}$ and see what happens.

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ (\mathbf{v}e^{rt})' &= A\mathbf{v}e^{rt} \\ \mathbf{v}re^{rt} &= A\mathbf{v}e^{rt} \\ r\mathbf{v} &= A\mathbf{v} \end{aligned}$$

But, $A\mathbf{v} = r\mathbf{v}$ means r must be an eigenvalue of A and $\mathbf{v}$ is a corresponding eigenvector! Furthermore, it is a theorem from linear algebra that if A has n real, distinct eigenvalues, then the corresponding eigenvectors must be linearly independent.

Example. Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 45 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution.

$$\det(A - rI) = \det \begin{bmatrix} -16 - r & -6 \\ 45 & 17 - r \end{bmatrix} = r^2 - r - 2 = (r+1)(r-2).$$

Thus, the eigenvalues are -1 and 2. Next we find eigenvectors.

Let $r = -1$. Then $A - rI = \begin{bmatrix} -15 & -6 \\ 45 & 18 \end{bmatrix}$. Then we solve

$$\begin{bmatrix} -15 & -6 \\ 45 & 18 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two resulting linear equations are equivalent. Both lead to $a = -\frac{2}{5}b$. Thus $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ is an eigenvector. Hence,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-t}$$

is a solution.

Now let $r = 2$. Then $A - rI = \begin{bmatrix} -18 & -6 \\ 45 & 15 \end{bmatrix}$. Then we solve

$$\begin{bmatrix} -18 & -6 \\ 45 & 15 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two resulting linear equations are equivalent. Both lead $a = -b/3$. Thus, $\begin{bmatrix} -1//3 \end{bmatrix}$ is an eigenvector. Hence,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$$

is a solution.

You can use the Wronskian to check that they are linearly independent, but this is already clear. At $t = 0$ we recover the eigenvectors, which we know are linearly independent and the Wronskian of a pair of solutions must either be always zero or never zero. Hence the Wronskian here is never zero and we have a fundamental solution set. We can write that the **general solution** is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}.$$

Now, suppose we had been given initial conditions $x(0) = 1$ and $y(0) = 2$. We need to find values for C_1 and C_2.

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = C_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{0}.$$

Thus,

$$\begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Recall, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Thus,

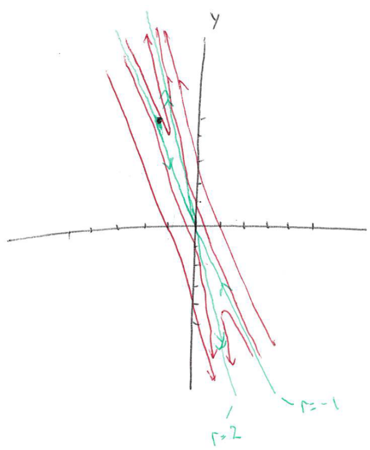
$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}.$$

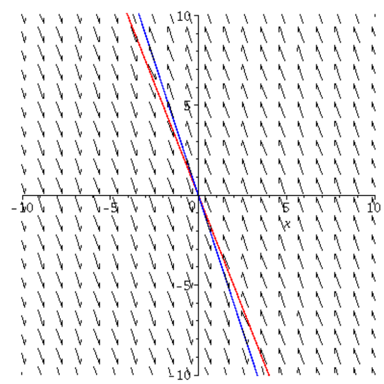
Therefore,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -5 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-t} + 9 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t} = \begin{bmatrix} 10e^{-t} - 9e^{2t} \\ \\ -25e^{-t} + 27e^{2t} \end{bmatrix}.$$

Next we will examine a rough sketch of the **phase portrait**. This is just a sketch of a lot of solution curves. First, if our initial condition was the origin would would stay there forever. It is called a **node**, a **rest point**, a **fixed point**, a **critical point**, or an **equilibrium**

point. (Anything with that many names, must be important!) Now if we had an initial condition on the eigenspace for $r = -1$, but not at the origin, the solution curve would just a a line tending toward the origin. For example if the initial condition were $x(0) = -2$ and $y(0) = 5$, the solution would be $\begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-t}$. Similarly, if we started on the eigenspace for $r = 2$, but not at the origin, the solution stay in this space, but the limit as t increases would to infinity. The limit as $t \to -\infty$ would be the origin. For other initial conditions the e^{-t} terms will tend toward zero, so solutions curves will be asymptotic to the $r = 2$ eigenspace as $t \to \infty$.But, they will be asymptotic to the $r = -1$ eigenspace as $t \to -\infty$. These insights allow us to sketch a rough phase portrait. See below left. On the right is a computer direction field with the two eigenspaces. Notice that the computer generated image would be hard to interpret if we did not understand how the system is supposed to behave.





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Some Observations. Any time we have one eigenvalue positive and the other negative, the phase portrait will behave similarly to the previous example. If both eigenvalues had been negative, all solutions would tend toward to origin as $t \to \infty$. If both eigenvalues had been positive, solution would move exponentially away from the origin when t increases, and would tend toward the origin as $t \to -\infty$. When all solutions near a point tend toward that point, the point is called an **attracting fixed point**. Your book calls such points **asymptotically stable nodes.** When all solutions near a point move away from that point, the point is called a **repelling fixed point.** When the solution curves behave as they did in the example above the fixed point is called a **saddle point**. Your book calls points where all or almost all solution curve move away from it **unstable nodes**.

Your textbook does more examples of these types. Next we study an example where one of the eigenvalues is zero.

Example. Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

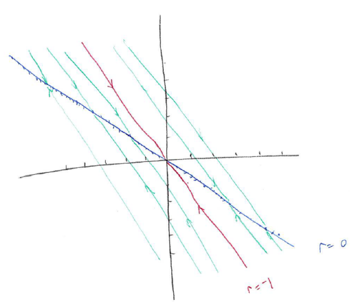
Solution. You can check the eigenvalues are 0 and -1. For 0 an eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. For -1 an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus, the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^0 + C^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} = \begin{bmatrix} -2C_1 + C_2 e^{-t} \\ C_1 - C_2 e^{-t} \end{bmatrix}.$$

As in the first example, a solution curve that starts in the eigenspace for -1 will tend toward the origin. But, lets pick an initial condition of the eigenspace for 0, say $x(0) = 4$ and $y(0) = -2$. Then, you can check, $C_1 = -2$ and $C_2 = 0$. Thus, the solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Nothing happens! The entire eigenspace for 0 is frozen in time. Every point is a rest point. What about initial conditions not on either eigenspace? As $t \to \infty$ they converge to $C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, which is some point on the eigenspace for 0. Below is a rough sketch of the phase portrait.



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Example. Our final example will be a 3×3 system. Solve

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & -10 & 2 \\ 4 & 10 & -2 \\ 4 & 8 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}.$$

Solution. First we find the eigenvalues.

$$\det\begin{bmatrix} -3-r & -10 & 2 \\ 4 & 10-r & -2 \\ 4 & 8 & -1-r \end{bmatrix} = \det\begin{bmatrix} 1-r & -r & 0 \\ 4 & 10-r & -2 \\ 4 & 8 & -1-r \end{bmatrix}$$

$= (1-r)((10-r)(-1-r)+16) = (1-r)(r^2-5r+6) = (1-r)(r-2)(r-3)$. Thus, the eigenvalues are 1, 2 and 3. (I used a row operation to simplify the matrix.) They are real and distinct. They are all positive, so we already know that all solutions move away from the origin (unless we started right at the origin) and will converge to the origin as $t \to -\infty$.

Next we find an eigenvector for each eigenvalue. Let $r = 1$. We must find a nonzero solution to

$$\begin{bmatrix} -4 & -10 & 2 \\ 4 & 9 & -2 \\ 4 & 8 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can use row operations to reduce the matrix

$$\begin{bmatrix} -4 & -10 & 2 \\ 4 & 9 & -2 \\ 4 & 8 & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & 10 & -2 \\ 0 & -1 & -0 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $a = c/2$, and $b = 0$. We treat c as a free parameter, and let $c = 2$. Thus, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvector.

Let $r = 2$. Now we need a nonzero solution to

$$\begin{bmatrix} -5 & -10 & 2 \\ 4 & 8 & -2 \\ 4 & 8 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Use row operations to convert this to

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $c = 0$ and $a = -2b$. We will use $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ as our eigenvector.

Let $r = 3$. We need a nonzero solution to

$$\begin{bmatrix} -6 & -10 & 2 \\ 4 & 7 & -2 \\ 4 & 8 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Use row operations to convert this to

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $a = -3c$, $b = 2c$ and c is free. We will use $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ as our eigenvector.

The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^t + C_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} e^{3t}.$$

Let's suppose we had been given an initial condition, say

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, we need to solve

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

You can find the inverse of the matrix, but it is easier to just use row operations to convert this problem to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}.$$

Thus, $C_1 = -1$, $C_2 = -4$ and $C_3 = 2$. We can now report that the solution for the given initial condition is

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -e^t + 8e^{2t} - 6e^{3t} \\ -4e^{2t} + 4e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix}.$$

$\square$