

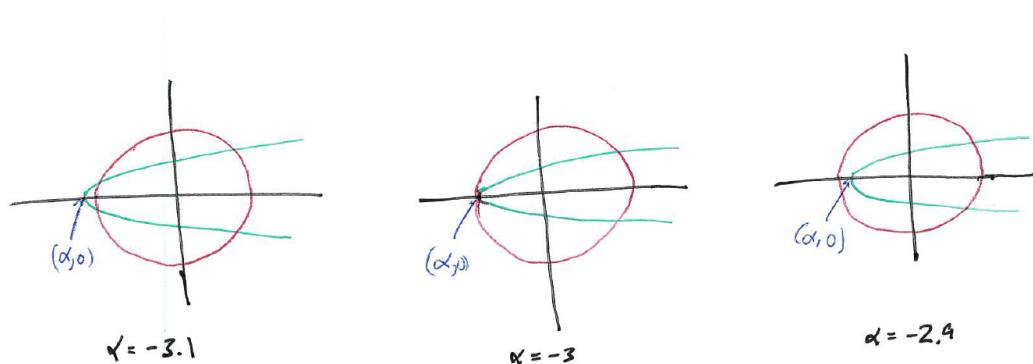
A Bifurcation Example

We are going to modify Example 2 from the lecture notes for Section 9.3 by adding a parameter α . Consider

$$\begin{aligned}x' &= x^2 + y^2 - 9, \\y' &= 3y^2 - x + \alpha.\end{aligned}$$

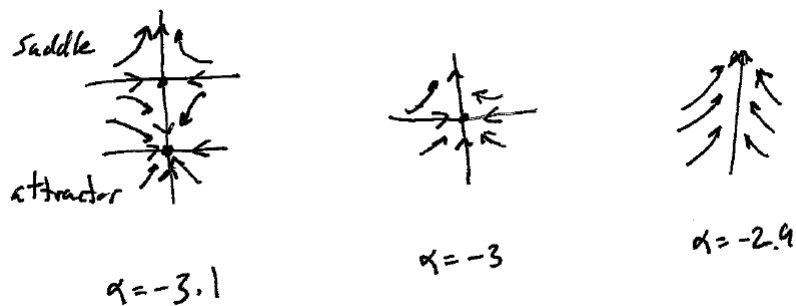
When $\alpha = -3$ we just have Example 2 and we know $(0, -3)$ is a critical point. Recall the linearization had eigenvalues -6 and 0 . But, numerically we found that that vector field was only $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $(0, -3)$. Just above this point solutions curve move upward, away from the critical point, and just below the critical points solution curve move upward toward $(0, -3)$, asymptotically.

Now, if $\alpha = -2.9$, then there would be no critical point at or near $(0, -3)$; the other two critical points would shift their position a bit. If $\alpha = -3.1$, then there would be a total of four critical points as can be seen in the figure below. The red circle is the set of points where $x' = 0$ as does not change with α . The green parabola is the set of points where $y' = 0$; it is given by $x = 3y^2 + \alpha$. Its vertex is at $(\alpha, 0)$. From picture we can see the number of critical points is four when $\alpha = -3.1$, three when $\alpha = -3$ and just two when $\alpha = -2.9$. The number of intersection points is just the number of critical points.

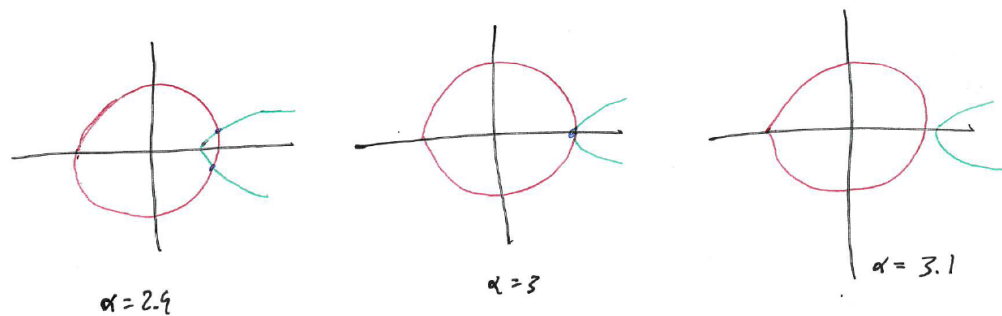


This phenomena is called a **bifurcation**. We often say that the *qualitative behavior* of the system changes when a parameter passes through a bifurcation value. In this example $\alpha = -3$ is a bifurcation value. We will not give a formal mathematical definition of a bifurcation, but the fact the linearization matrix had a zero eigenvalue is typical of bifurcations and is the fact that the green and red curves meet *tangentially* rather than *transversely*.

Let's look at little more closely at this bifurcation. For $\alpha = -3.1$ you can do the linearization at the two critical points, or just study the vector field nearby them, and see the upper one is a saddle and lower one is an attractor. As α gradually moves to -3 these two critical points will get closer and closer and at $\alpha = -3$ they "merge" into one critical point. As α increases above -3 this critical point vanishes. Poof! See below. In fact, this is an example of a **saddle-node bifurcation**.

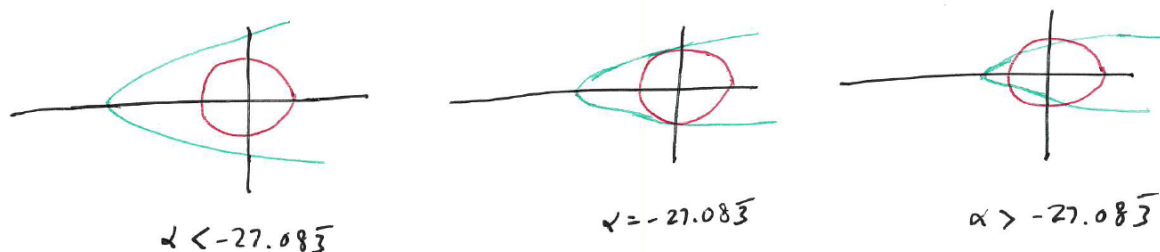


Our system will undergo another bifurcation when $\alpha = 3$. At this value the remaining two critical points merge and then vanish. See below.



Students should check that the linearization matrix at the critical point $(3,0)$ has a zero eigenvalue. Study the vector field just before the bifurcation, at the bifurcation value and just after.

There is one other value of α for which we get a bifurcation. Pictorially, you can see it below for what must be a large negative value of α .



We can compute the value of α where this bifurcation occurs. When $y' = 0$ we have $y^2 = (x - \alpha)/3$. When $x' = 0$ we have $x^2 + y^2 - 9 = 0$. Thus, when both are zero we have

$$x^2 + \frac{x - \alpha}{3} - 9 = 0.$$

Therefore,

$$3x^2 + x - (\alpha + 27) = 0.$$

By the quadratic formula

$$x = \frac{-1 \pm \sqrt{1 + 12(\alpha + 27)}}{6}.$$

There are no real solutions when $1 + 12(\alpha + 27) < 0$. There is exactly one solution when $1 + 12(\alpha + 27) = 0$, namely $x = -1/6$. There are two real solutions for x when $1 + 12(\alpha + 27) > 0$. Thus, we can find the bifurcation value for α by solving

$$1 + 12(\alpha + 27) = 0.$$

This gives $\alpha = -\frac{325}{12} = -27.08\bar{3}$.

I made an animation of the system where you can see some of the bifurcations happen. See the link on the webpage.