

Course policies for calculus B-20, Winter quarter.

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Bulletin Board: I have set up a computer bulletin board for the class. It is called "nwu.school.cas.math.b-20". You may also want to use "nwu.school.cas.math.calculus". The news reader I use is "trn". This is an experiment, we'll see how it goes.

Office Hours: M 11-12, W 1-2, F 1-2, and by appointment.

Texts:

1. *Calculus with Analytic Geometry* Edwards & Penny; 15.8, 15.9 and all of 16.
2. *Elementary Differential Equations*, Boyce & DiPrima; chapters 1 to 3.
3. *A Brief Course in Linear Algebra*, Evans, sections 1.1 to 1.6.

Tests: There will be two mid-term tests and a final. The mid-terms will count 100 points each and the final will be worth 200 points. Parts of these may be take home. For the in class portions, you can use a sheet of notes and a calculator, but you must show your work to get credit.

If you miss a test because you were deathly ill and have a note from a doctor, or because a close relative died and you can give me the funeral home's phone number to check, then you may take a make-up test. Otherwise, forget it. Also, you must call or e-mail me in advance.

The first test will be on Thursday, Feb. 2. It will cover the end of chapter 15 and (hopefully) all of 16. The second test will be on Thursday, Feb. 23. It will cover chapters 1 through 3 of the differential equations book. The final is from 12-2, Monday, March 13.

Homework: You are expected to do the homework, but it will not be graded. You may hand it in if you want feed back on your work. I encourage you to hand in at least some of your work.

Thursday meetings: You will meet with a teaching assistant on Thursdays at noon to discuss homework problems. You may be asked to present your work. If you don't go you will have to serve time in purgatory. For example, I could tell the teaching assistant to start giving quizzes.

Extra Credit (yea!): You can earn up to 50 points extra credit by doing projects. The little project sections in your calculus text may help you find ideas. They would count for maybe 5 points. Also, I will try to assign extra credit problems in class.

Cheating: The minimum penalty for cheating is a grade of F in the course. Your Dean may wish to take further action. This is a big no no.

Practice Test 1: B-20-2

1. Evaluate $\int_C x^2 dx + y^2 dy$ where C is the polygonal path running from (1,1) to (1,2) to (2,3).

2. Let

$$\mathbf{F} = \left\langle x + \frac{x}{(x^2 + y^2 + z^2)^{2/3}}, \frac{y}{(x^2 + y^2 + z^2)^{2/3}}, \frac{z}{(x^2 + y^2 + z^2)^{2/3}} \right\rangle.$$



Let the surface S be the unit sphere centered at the origin. Find $\int \int_S \mathbf{F} \cdot \mathbf{n} dS$.

3. Let S be the lower hemisphere $x^2 + y^2 + z^2 = 4$, $z \leq 0$ with unit normal \mathbf{n} pointing away from the origin. Let

$$\mathbf{F} = \langle ze^{x-y}, ze^{y-x}, xyz \rangle.$$

Find $\int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$.

4. Show that $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl} \mathbf{F} - \mathbf{F} \cdot \text{curl} \mathbf{G}$.

5. Find the surface area of the part of the surface $z = y^2 - x^2$ that is inside the cylinder $x^2 + y^2 = 4$.

Notes: We did not have time to cover the subsection on conservative force fields on page 924, but it is important. We did not have time to cover Example 4 on page 926, but it is important.

Test 1

B-20

Winter Quarter, 1994

Name: _____
Student Number: _____

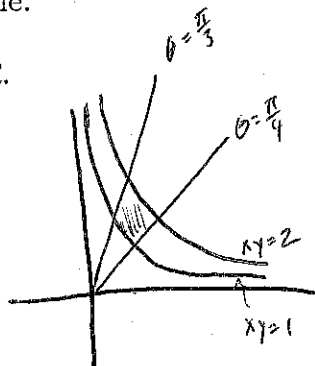
Problems 1 through 5 are worth 15 points each.
You will be given a take home problem in class tomorrow. It will be worth 25 points.

1. Let R be the region bounded by the four curves:

$$xy = 1, \quad xy = 2, \quad \theta = \frac{\pi}{4}, \quad \theta = \frac{\pi}{3},$$

where θ is the polar angle.

(a) (5 points.) Draw R .



(I should have said, "in the 1st quadrant.")

(b) (10 points.) Let $u = xy$ and let T be the transform from (u, θ) -space to (x, y) -space. Find J_T . Hint: To find T notice that $\theta = \text{a constant}$ is a line $y = mx$ with $m = \tan \theta$. Recall: $d \tan \theta / d\theta = \sec^2 \theta$.

$$J_T = \begin{vmatrix} x_u(u, \theta) & x_\theta(u, \theta) \\ y_u(u, \theta) & y_\theta(u, \theta) \end{vmatrix}$$

Given u & θ what are x & y ?

$$y = x \tan \theta \Rightarrow u = x^2 \tan \theta \Rightarrow x = \sqrt{\frac{u}{\tan \theta}}$$

$$\text{Then } y = \tan \theta \sqrt{\frac{u}{\tan \theta}} = \sqrt{u \tan \theta}$$

$$\text{Thus } J_T = \begin{vmatrix} \frac{1}{2} \frac{1}{\sqrt{u \tan \theta}} & -\frac{1}{2} \sqrt{\frac{u}{\tan^3 \theta}} \sec^2 \theta \\ \frac{1}{2} \sqrt{\frac{\tan \theta}{u}} & \frac{1}{2} \sqrt{\frac{u}{\tan \theta}} \sec^2 \theta \end{vmatrix} =$$

$$\frac{1}{4} \frac{\sec^2 \theta}{\tan \theta} - \frac{1}{4} \frac{\sec^2 \theta}{\tan \theta} = \frac{1}{2} \frac{\sec^2 \theta}{\tan \theta} =$$

$$\frac{1}{2} \frac{\frac{1}{\cos^2 \theta}}{\frac{\sin \theta}{\cos \theta}} = \frac{1}{2 \sin \theta \cos \theta} = \frac{1}{\sin 2\theta}$$

2. Let $F(x, y) = \left\langle \frac{x^2}{(x^2+y^2)^{\frac{5}{2}}}, \frac{y^2}{(x^2+y^2)^{\frac{5}{2}}} \right\rangle$.

- (a) (10 points.) Compute $\int_C F \cdot dr$ where, C is the unit circle running from $(1, 0)$ to $(0, 1)$.

On the unit circle $x^2 + y^2 = 1$.



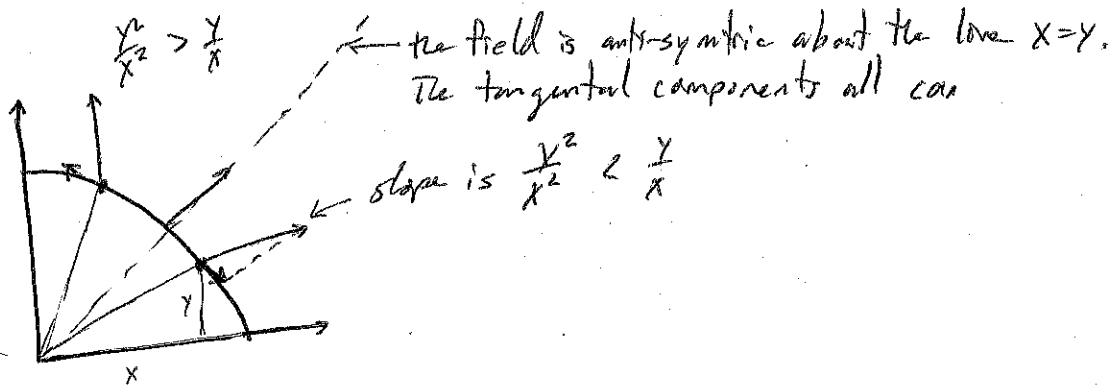
$$\begin{aligned} x &= \cos t \\ y &= \sin t \quad t \in [0, \frac{\pi}{2}] \\ x' &= -\sin t \\ y' &= \cos t \end{aligned}$$

$$\int_C F \cdot dr = \int_C x^2 dx + \int_C y^2 dy$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t d(\cos t) + \int_0^{\frac{\pi}{2}} \sin^2 t d(\sin t)$$

$$\frac{1}{3} \cos^3 t \Big|_0^{\frac{\pi}{2}} + \frac{1}{3} \sin^3 t \Big|_0^{\frac{\pi}{2}} = \frac{1}{3}(0-1) + \frac{1}{3}(1-0) = 0.$$

- (b) (5 points.) In part (a) your answer was (positive/negative/zero) [pick one]. Why? Hint: draw the vector field in the first quadrant.



3. Let S be a sphere of radius r centered at the origin. Let

$$\mathbf{F} = \langle a_1 + b_1x + c_1x^2, a_2 + b_2y + c_2y^2, a_3 + b_3z + c_3z^2 \rangle.$$

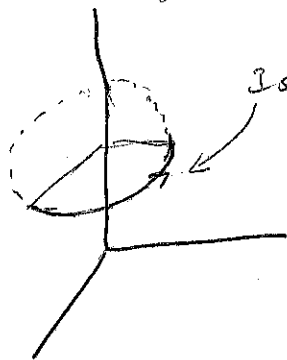
Show that the flux through S is $(b_1 + b_2 + b_3) \frac{4}{3} \pi r^3$. Hints: Use the divergence theorem. You can save some time by noticing that $\iiint_V x dV = 0$ since the centroid of a uniform solid sphere is the origin. (V is the solid sphere inside S .)

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{F} dV$$

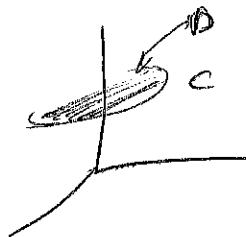
$$\nabla \cdot \mathbf{F} = b_1 + b_2 + b_3 + (a_1 + a_2 + a_3)x.$$

$$\begin{aligned} \iiint_V \nabla \cdot \mathbf{F} dV &= \iiint_V (b_1 + b_2 + b_3) dV + \iiint_V (a_1 + a_2 + a_3)x dV \\ &= (b_1 + b_2 + b_3) \iiint_V dV + (a_1 + a_2 + a_3) \iiint_V x dV \\ &= (b_1 + b_2 + b_3) \frac{4}{3} \pi r^3 \end{aligned}$$

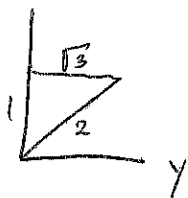
4. Let $\mathbf{F} = \langle y, 2x - z, z - x \rangle$ and let C be the curve in \mathbb{R}^3 given by the intersection of $x^2 + y^2 + z^2 = 4$ and $z = 1$. Find



I should have said "counter-clockwise" $\oint_C \mathbf{F} \cdot \mathbf{T} ds$.



So, our circle C of radius $\sqrt{3}$ bounds a disk D of area 3π .



$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA$$

$$\mathbf{n} = \langle 0, 0, 1 \rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & 2x-z & z-x \end{vmatrix} = \langle \quad, \quad, 2-1 \rangle$$

don't matter!

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = 1.$$

$$\iint_D 1 \, dA = \boxed{3\pi}.$$

5. Let $\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ be a twice differentiable vector field. Let S be a parametric surface that bounds a region V . Show that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0.$$

[Show all steps.]

Use divergence thm.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iiint_V \nabla \cdot (\nabla \times \mathbf{F}) dV.$$

But $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, as done in class.
(see your class notes.)

Take home problem for test 1.

Due 12 noon, Monday, February 6.

25 points. Show all work and copy your work over neatly. Staple pages together. You are not to discuss this with others or use books other than your textbook. You may use your class notes.

On separate sheets of graph paper, carefully draw the vector fields

$$\langle x, y \rangle, \langle -x, y \rangle, \langle y, x \rangle, \langle y, -x \rangle.$$

For each vector field compute the following limit

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \oint_{C_r} \mathbf{F} \cdot \mathbf{T} ds,$$

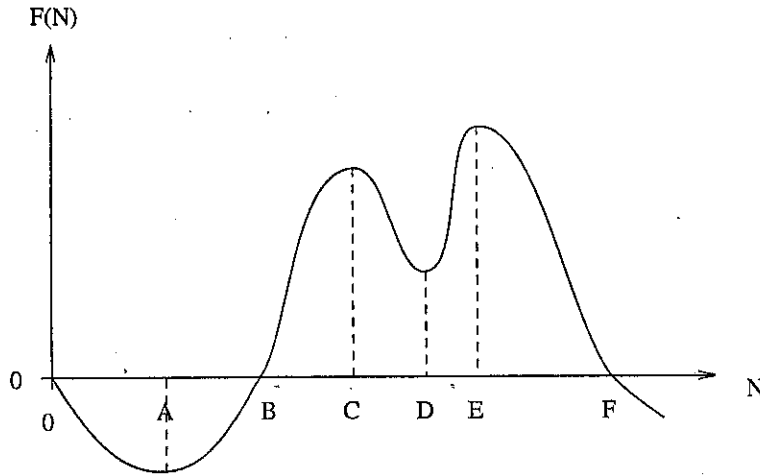
where C_r is a circle of radius r and center $(0,0)$.

For each vector field compute $Q_x - P_y$. Discuss the significance of your findings.

too easy
don't reuse

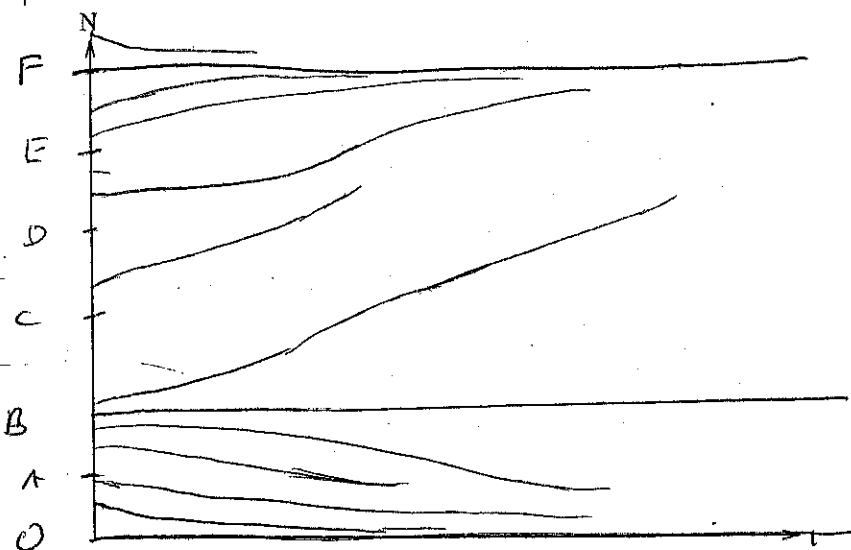
Practice Test 2: B-20, Winter Quarter, 1994

1. Suppose $N'(t) = F(N(t))$, where the graph of $F(N)$ is given below. Carefully draw the integral curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $N(t)$ and t are nonnegative.



Concavity

⊕
⊖
⊕
⊖
⊕
⊖
⊕
⊖
⊕
⊖



stable equilibrium

unstable eq.

stable eq (at least from above)

2. Find the general solution of the differential equation $y' + y \sin x = \sin x$.
3. Find the general solution of the differential equation $(e^x + y')y'' + e^x y' = \sin x$. Do this by letting $v = y'$. Express $y(x)$ as an integral. Do not evaluate it.
4. Solve the initial value problem $y'' - 4y' + 4y = 0$, where $y(0) = 3$ and $y'(0) = 1$.
5. Find a particular solution of $y'' + 3y' - 5y = \cos x$.

2.

$$\frac{y'}{1-y} = \sin x \quad \text{integrate to get } -\ln|1-y| = -\cos x + C$$

$$\text{or } 1-y = Ce^{\cos x} \Rightarrow y(x) = 1 - Ce^{\cos x}$$

3.

Let $v=y$!

$$\underbrace{(e^x + v)}_{N(x,v)} v' + \underbrace{(e^x v - \sin x)}_{M(x,v)} = 0$$

It is exact. $N_x = M_v = e^x$.

$$\psi = \int M dx = e^x v + \cos x + h(v).$$

$$\psi_v = e^x + h'(v) = N = e^x + v$$

$$\Rightarrow h'(v) = v$$

$$h(v) = \frac{1}{2}v^2 + C$$

Thus $\psi(x,v) = \frac{1}{2}v^2 + e^x v + \cos x = C$ is a solution.Solving for v gives:

$$v = -e^x \pm \sqrt{e^{2x} - 2(\cos x - C)}$$

$$y = \int v dx + C_2.$$

4.

$$y = 3e^{2x} - 5xe^{2x}$$

5.

$$y = \frac{1}{5} \sin x - \frac{2}{5} \cos x.$$

Take home problem for test 2.

Due 12 noon, Monday, March 2.

25 points. Show all work and copy your work over neatly. Staple pages together. You are not to discuss this with others or use books other than your textbook. You may use your class notes.

Problem: On the test you found that the general solution of $2x(y - 3) + (x^2 + y)y' = 0$, (*), was $2x^2(y - 3) + y^2 = C$, (#).

The goal of this problem is gain a complete understanding of the integral curves of (*).

1. Show that $y(x) = 3$ is the solution of the initial value problem (*) with $y(0) = 3$.
2. Solve (#) for y using the quadratic formula. In what follows pay close attention to which branch (+ or -) you are on.
3. Let $k > 3$ be fixed. Find the solution of (*) with initial condition $y(0) = k$ (try this for $k = 4$ first). What happens to $y(x)$ as $x \rightarrow \pm\infty$?
4. Now let $0 < k < 3$. Repeat step 3 (but try $k=1$ first).
5. Rewrite (*) as

$$y' = \frac{2x(3 - y)}{x^2 + y}.$$

What happens to y' along the parabola $y = -x^2$? We shall call this curve the *exceptional parabola*.

6. Let $k > 0$ be fixed. Solve (*) with $y(k) = 0$ (try $k = 1$ first). What happens to $y(x)$ as $x \rightarrow \infty$? Over what interval is this solution curve defined?
7. Repeat step 6 for $k < 0$.

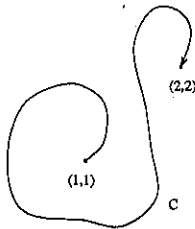
You should now be able to get a complete picture of the integral curves of (*) above the exceptional parabola. Do this! The analysis below the exceptional parabola is more subtle. It will be helpful to divide this area into two parts. Let $g(x) = -x^2 - \sqrt{x^4 + 6x^2}$. Let region **I** be given by points (x, y) such that $-x^2 > y > g(x)$ and region **II** be given by points (x, y) such that $y < g(x)$. See the figure.

8. Let $(x_0, y_0) \in \mathbf{II}$. For example $(0, -1) \in \mathbf{II}$. Solve (*) with $y(x_0) = y_0$. Show that the solution remains in **II** and that it is asymptotic to $g(x)$ as $x \rightarrow \pm\infty$.
9. Let (x_0, y_0) be such that $g(x_0) = y_0$, but not $(0, 0)$. For example $g(1) = -1 - \sqrt{7}$. Show that $y(x) = g(x)$ solves (*) with $y(x_0) = y_0$. On what interval is it valid? (Your answer will depend on whether x_0 is positive or negative.)
10. Let $(x_0, y_0) \in \mathbf{I}$. For example $(1, -2) \in \mathbf{I}$. Show that the solution of (*) with $y(x_0) = y_0$ stays in **I** and is asymptotic to $g(x)$ for $x \rightarrow \infty$ OR $x \rightarrow -\infty$, but NOT both.

You now have all the information you need to give a complete picture of the integral curves of (*). Do this! Also fill in the table below.

Practice Final, B-20, Winter Quarter, 1994

1. a) Show that $F(x, y) = \langle 3x^2y + y^2, x^3 + 2yx \rangle$ is path independent. b) Compute $\int_C F \cdot T ds$, where C is the curve below.



2. Find the surface integral of $F(x, y, z) = 3x + 2y + z$ over the surface defined by the portion of the plane $x + 2y - z = 1$ inside the cylinder $x^2 + y^2 = 1$.
3. Let T be the solid bounded by the three coordinate planes and the plane $x + y + z = 1$. Let S be the boundary of T . Let \mathbf{n} be the outward pointing unit normal vector. Let $\mathbf{v} = \langle x + xz, y - yz, z + xy \rangle$. Evaluate the flux $\int_S \mathbf{v} \cdot \mathbf{n} dS$.
4. Let $f(x, y)$ and $g(x, y)$ be twice differential^{able} scalar functions of two variables. a) Show that $\nabla \cdot (\nabla f) = \partial^2 f / \partial x^2 + \partial^2 f / \partial y^2$. b) Show that $\nabla \cdot (\nabla fg) = f \nabla \cdot (\nabla g) + g \nabla \cdot (\nabla f) + 2 \nabla f \cdot \nabla g$.
5. Find the general solution of $y'' + 4y' + y = e^{2x} + \cos x$.
6. Sketch the direction field of $y' = 3x + 2y$.
7. Show there exists two linearly independent solutions to $y'' + p(x)y' + q(x)y = 0$. Assume that every initial value problem has a unique solution.
8. The population $P(t)$ of a country is determined by an intrinsic growth rate $r =$ number of births per 100,000 people minus the number of deaths per 100,000 people, and $M(t)$, a migration function (number people who came in minus the number of people who left).
- a) Assuming r and M are constant, set up and solve the differential equation for $P(t)$. Assume $P(0) = P_0$.
- b) Keep r constant but suppose $M(t) = \cos t$. Hint: $\int e^{au} \cos bu du = \frac{e^{au}}{a^2 + b^2} (a \cos bu + b \sin bu) + C$.
9. Find the inverse of

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

10. For which values of α does the following system have a non-trivial (x and y not both zero) solution?

$$\begin{bmatrix} 1 & \alpha \\ \alpha & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Final Exam

B-20

Winter Quarter, 1994

Name: _____

Student Number: _____

There are 10 problems worth 20 points each.

1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	
Total	

2. Find the surface area of the paraboloid $z = x^2 + y^2 - 9$ below the xy -plane.

4. Let $\mathbf{F}(x, y, z) = -k\mathbf{r}/r^3$, where $\mathbf{r} = \langle x, y, z \rangle$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

(a) (15 points) Find the flux of \mathbf{F} through S .

(b) (5 points) Explain why the divergence theorem cannot be applied in this case.

6. Let f and g be solutions of $y'' + p(x)y' + q(x)y = 0$. Show that

$$W(f, g)(x) = ce^{-\int p(x)dx}.$$

8. Sketch the direction field of $y' = 2x - y$. Draw a few of the integral curves. Do not solve the equation.

10. Prove that a 2×2 system of linear equations,

$$ax + by = e$$

$$cx + dy = f,$$

has a unique solution if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$