Ch 2, Sec 8 + parts of Sec 9

Cantor Sets

Def: A metric space is called a Cantor space if it is nonempty, compact, perfect, and totally disconnected.

We defined compact and perfect before. A space is totally disconnected if the only nonempty connected subsets are one-point sets. Thus, $\mathbb{Z}$ and $\mathbb{Q}$ are examples. There are several equivalent definitions. The textbook gives one on page 105. Show that it is equivalent to the one given here. (This is harder than I thought it was. One direction is easy. For the other see Willard's Gen. Top. ~~30F~~ 29D.)

Thm: Cantor spaces exist!

pf You have likely seen the cannonical example, the middle thirds Cantor set in $\mathbb{R}$.

Let $C^0 = [0,1]$, $C^1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$,

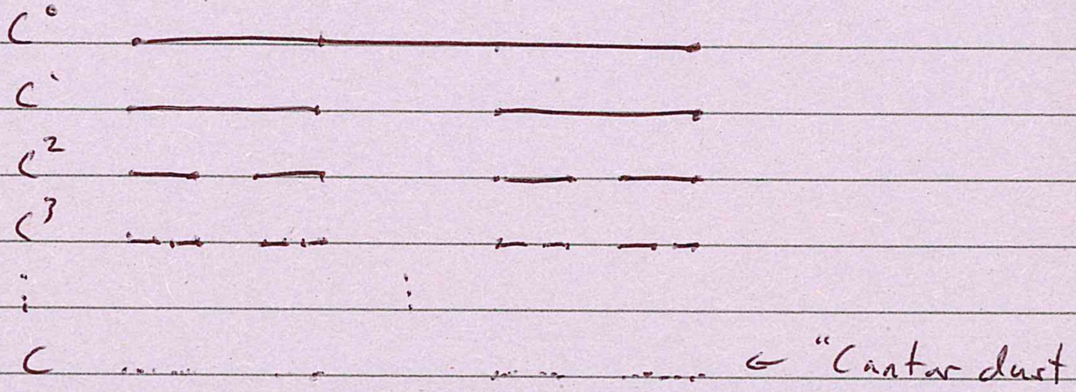
$C^2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$, etc.

Each C^{n+1} is constructed from C^n by removing the open middle third intervals from each of the 2^n segments in C_n.

Thus, $C^{n+1} = C^n - \bigcup_{i \text{ odd}} \left(\frac{i}{3^{n+1}}, \frac{i+1}{3^{n+1}}\right)$.

You should work out the next couple of levels, C^3, C^4.

Finally, $C = \bigcap_{n=1}^{\infty} C^n$, is called the *Cantor set*, or the *middle thirds Cantor set*. Here is a "picture."



Now we prove it is a Cantor space. It is a metric space as a subspace of $\mathbb{R}$. As the nested intersection of nonempty compact sets it must be nonempty and compact by Thm 34 on page 82. Although we can see directly that $0, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}$, and 1 etc, are members of C.

Next we show that C is perfect. For each n name the segments in C^n by $C^{n1}, C^{n2}, \dots, C^{n2^n}$. Let $p \in C$. Then $p \in C^n$ for each n. Let $n(p)$ be the index s.t. $p \in C^{n\,n(p)}$ for each n.

We want to show that p is a cluster point of C. Let $\varepsilon > 0$ and consider $(p-\varepsilon, p+\varepsilon)$. For large enough values of n (how large?), $C_n^{n, n(p)} \subset (p-\varepsilon, p+\varepsilon)$, and each $C^{n, n(p)} \cap C$ is an infinite set since it contains all the end points. Thus $C \cap (p-\varepsilon, p+\varepsilon)$ is an infinite set. Thus each $p \in C$ is a cluster point. Since C is closed there are no cluster points not in C. Thus C is perfect.

Lastly, we show C is totally disconnected. The connected subsets of $\mathbb{R}$ are intervals and one-point sets. Every interval contains it interior and we need not consider unbounded intervals since $C \subset [0,1]$. Suppose $(a,b) \subset C$. But, for n large enough (how large?) ~~it~~ (a,b) will contain one of the open segments we are removing. Thus $(a,b) \not\subset C$. Thus the only (nonempty) connected subsets of C are the one-point sets. Hence, C is totally disconnected.

We conclude that C is a Cantor space. $\boxtimes$

Thm Every Cantor space is uncountable.

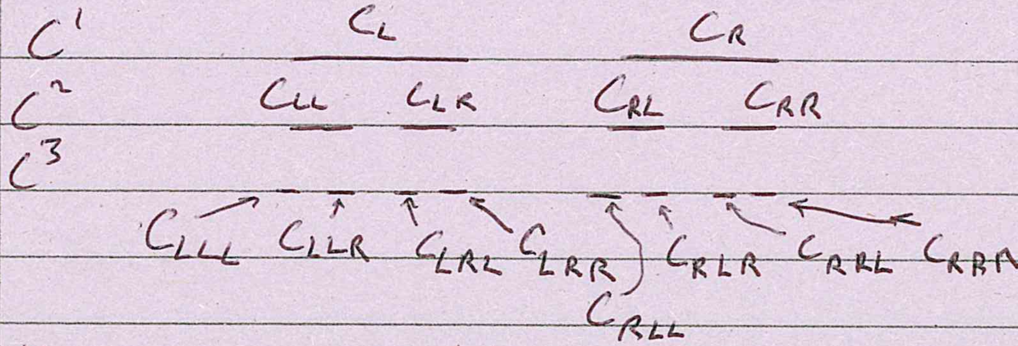
Pf By Thm 56, page 94, we know that every nonempty, perfect, complete metric space is uncountable. The textbook claims compact spaces are complete, which finishes the proof. However, I could not find where Pugh showed this, but it is easy to check.

Let K be a compact space and let (a_n) be a Cauchy seq in K. By compactness it has a convergent subseq. Suppose $a_{n_k} \to p \in K$. You can use the triangle ineq. to show $a_n \to p$ as well. Thus, K is complete. $\square$

It is instructive however to do a direct proof that the middle thirds Cantor set is uncountable. The method we will develop can be used to show an amazing fact: $\exists$ a continuous onto function $f: C \to [0,1]$.

We use a different labeling of the segments that make up each C^n. This in turn will yield an "address" for each $p \in C$ that will be an infinite string of the symbols L and R.

Let $C_L = [0, \frac{1}{3}]$ and $C_R = [\frac{2}{3}, 1]$. L is for left, R is for right. Next let $C_{LL} = [0, \frac{1}{9}]$, $C_{LR} = [\frac{2}{9}, \frac{1}{3}]$, $C_{RL} = [\frac{2}{3}, \frac{7}{9}]$ and $C_{LL} = [8/9, 1]$. Here is a picture:



You see the pattern it trust.

Each C^n has 2^n segments. Each will be labeled with a string of n symbols, each an L or an R. In fact they are listed alphabetically. For C^4 we have:

LLLL	LRRR	RRRL
LLLR	RLLL	RRRR
LLRL	RLLR	
LLRR	RLRL	
LRLL	RLRR	
LRLR	RRLL	
LRRL	RRLR	

Let $p \in C$. Then we will associate to p an "address," $w(p) = (w_i)_{i=1}^{\infty}$, each $w_i = L$ or R. We do this as follows.

Let w be an infinite sequence of L's and R's. Let $w|n = (w_1, w_2, \ldots w_n)$. This is called the truncation of w after the first n terms.

Let $w(p) = (w_i)_{i=1}^{\infty}$ where $p \in C_{w|n}$ $\forall n$. This is the address of p.

Addresses are unique and every possible seq is used as an address of some $p \in C$. The second statement is clear. Suppose $p, q \in C$. For n big enough (how big?) p and q will not be in the same segments of C^n. Thus they won't have the same address.

Now, define $f: C \to [0,1]$ as follows.

$$f(p) = s = 0.s_1 s_2 s_3 s_4 s_5 s_6 \ldots$$

$\uparrow$ zero

where $s_i = 0$ if $w_i = L$ and $s_i = 1$ if $w_i = R$. Now regard $s = 0.s_1 s_2 s_3 \ldots$ as the base 2 representation of a real number. We get each number in $[0,1)$, but also 1 since $1 = 0.1111\ldots$, using base 2. Since f is onto C is uncountable.

But f is also continuous! To prove this we'll use the fact that f is cont. iff it takes convergent sequences, to conv. seq's.

First, instead of using L's and R's in the addresses, replace the L's with 0's and the R's with 2's. Now watch! If $w(p) = (w_1, w_2, w_3, \ldots)$ then $p = 0.w_1w_2w_3\ldots$ regarded as the base 3 expansion of p. That is

$$p = \sum_{i=1}^{\infty} \frac{w_i}{3^i}.$$

You'll need to think about that for a while. The elements of C are exactly those points in $[0,1]$ whose ternary expansion can be given with only 0's and 2's. Remember $1/3 = 0.1$ (base 3) $= 0.0222\ldots$ (base 3), $7/9 = 0.21$ (base 3) $= 0.20222\ldots$ (base 3) etc. Play with this.

Suppose (p_n) is a seq in C and $p_n \to p \in C$. Let $a_n = f(p_n)$ and $a = f(p)$. We will show that $a_n \to a \in [0,1]$. This should "seem obvious" since we just replace 2's with 1's and switch to base 2. Here are the details.

Let $\varepsilon > 0$. We seek a natural number N s.t.

$$n \geq N \Rightarrow |a_n - a| < \varepsilon.$$

Write $p_n = \sum_{i=1}^{\infty} \frac{w_{ni}}{3^i}$ and $p = \sum_{i=1}^{\infty} \frac{w_i}{3^i}$, where each $w_{ni}, w_i = 0$ or 2. Then

$$a_n = f(p_n) = \sum_{i=1}^{\infty} \frac{w_{ni}/2}{2^i} \quad \text{and} \quad a = f(p) = \sum_{i=1}^{\infty} \frac{w_i/2}{2^i}.$$

Now a_n is closed to a if the first m terms agree. Suppose this is the case.

$$|a_n - a| \leq \frac{1}{2} \sum_{i=m+1}^{\infty} \frac{|w_{ni} - w_i|}{2^i} \leq \frac{1}{2} \sum_{i=m+1}^{\infty} \frac{2}{2^i} = \sum_{i=m+1}^{\infty} \frac{1}{2^i}$$

$$= \sum_{i=0}^{\infty} \frac{1}{2^i} - \sum_{i=0}^{m} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} - \frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^{m-1}.$$

We choose M big enough that $m \geq M \Rightarrow \left(\frac{1}{2}\right)^{M-1} < \varepsilon$. Then $|a_n - a| < \varepsilon$. But we have to relate this to the index n.

If the first M terms of a_n and a agree then so do the first M terms of p_n and p. We need to find N s.t. $n \geq N \Rightarrow$ the first M terms of p_n and p agree.

$\exists N$ s.t. $n \geq N \Rightarrow |p_n - p| < \left(\frac{1}{3}\right)^{M-1}$.

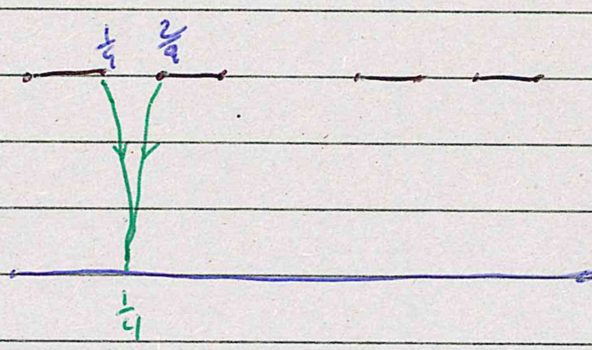
If the first term where p_n and p disagree is the k^{th} term then

$$|p_n - p| \geq \frac{1}{3^k}.$$

Thus if $|p_n - p| < \left(\frac{1}{3}\right)^{M-1}$, then p_n and p agree for the first M terms. Thus

$n \geq N \Rightarrow |a_n - a| < \varepsilon$. Thus $a_n \to a$.

I claim we can visually $\boxed{\text{see}}$ this function.



$f\left(\frac{1}{4}\right) = f(0.01) = f(0.00222\ldots) = 0.00111\ldots = 0.01$

$f\left(\frac{2}{9}\right) = f(0.02) = 0.01$

Another amazing fact is any two Cantor spaces are homeomorphic! See Thm 72 on page 112. The rough idea of the proof is...

Notice that C can be covered by 2^n open (actually clopen) disjoint sets for any n.

Example: 

You can do the same for any Cantor space. Thus you can create an addressing systems as we did for C. The sets of addresses can be put into one-to-one correspondence. This bijections will be a homeo. So every Cantor space is homeo to C. Done!

See also Willard's General Topology, section 30.

Thus $\frac{1}{9}$ and $\frac{2}{9}$ are identified; they are mapped to $\frac{1}{4}$.
Check out "end pt pairs."

$$f(\tfrac{1}{3}) = f(0.1) = f(0.0222\ldots) = 0.0111\ldots = 0.1 = \tfrac{1}{2}$$
$$f(\tfrac{2}{3}) = f(0.2) = 0.1 = \tfrac{1}{2}$$

All of the "gaps" in C are " ~~pinche~~ "pinched" together.
Check this for other "end pt pairs." Now you can <u>see</u> the behavior of f.

But even more is true. On page 108 is the <u>Cantor Surjection Thm</u>:

Given any compact, nonempty metric space M, there exist a continuous surjection of C onto M.

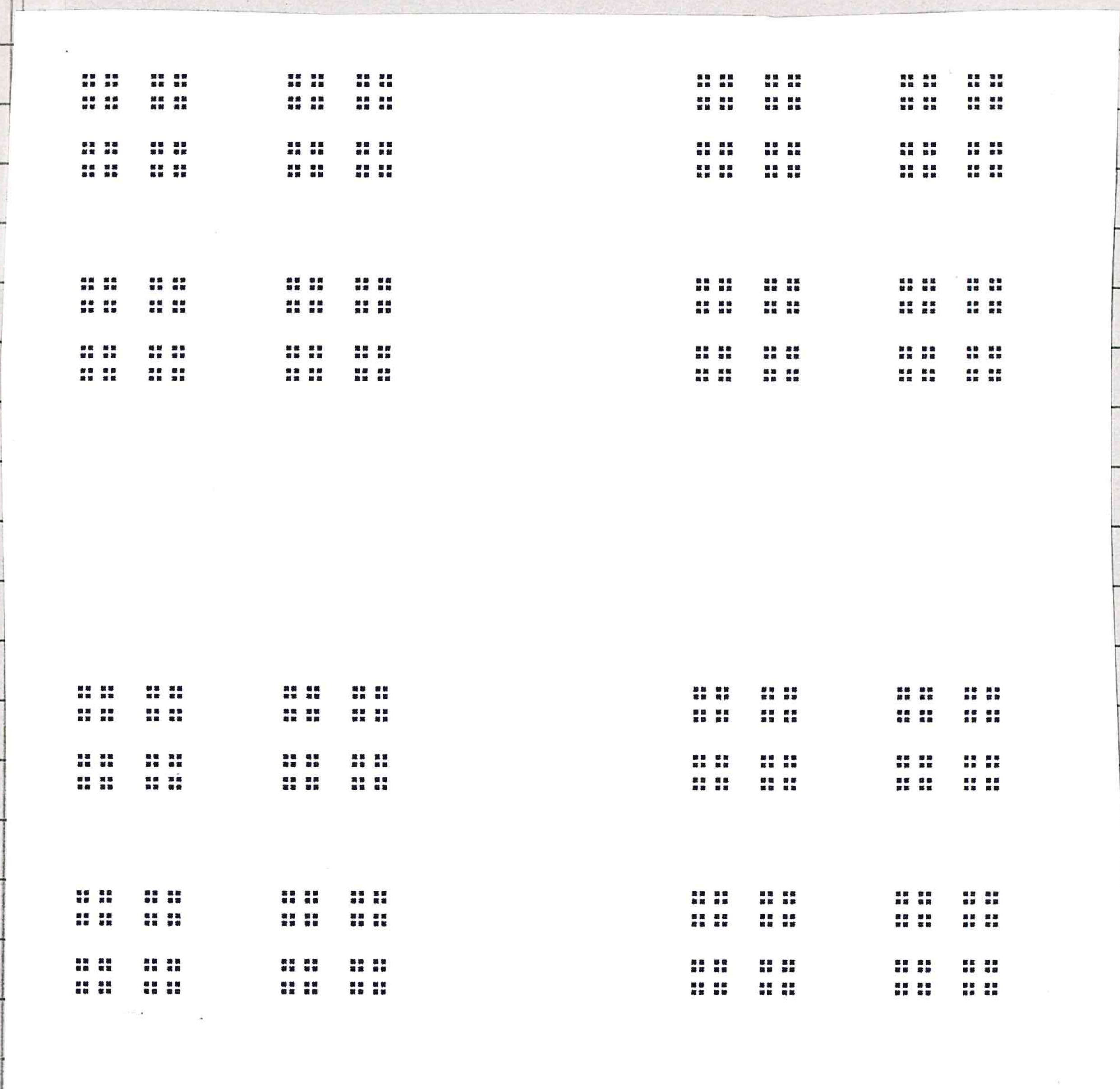
You can read the proof on your own.

Some other Cantor sets

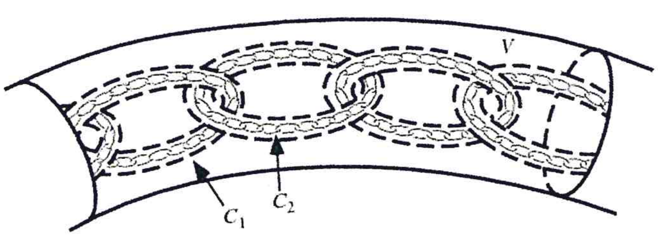
Instead of subtracting middle thirds we could subtract of middle segments of length $(\frac{1}{4})^n$ or $(\frac{1}{5})^n$, etc. These also satisfy the requirements to be a Cantor space: nonempty, compact, perfect ad totally disconnected.

Consider $C \times C$ as a subset of $\mathbb{R}^2$. It is also a Cantor space.

From Wikipedia



Another example is Antoine's necklace. You start with a solid torus in $\mathbb{R}^3$. Inside, make a chain of smaller solid tori. Inside each of these make even smaller chains of solid tori. Continue this and take their infinite intersection. The result is a Cantor set call Antoine's necklace.



From MathWorld.

How to measure the "length" of a Cantor set in $\mathbb{R}$.

1. For the middle thirds Cantor set the total length of C_n is $(\frac{2}{3})^n$. Check this. Then the limit $(\frac{2}{3})^n \to 0$. So, the "length" of C itself is zero. We say it has measure zero.

2. We do this for the middle fourths Cantor set. Let $F_0 = [0,1]$, $F_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$, $F_2 = [0 \frac{11}{64}] \cup$

$[\frac{13}{64}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{51}{64}] \cup [\frac{53}{64}, 1]$, etc. $F = \bigcap_{n=0}^{\infty} F_n$.

It is easier to work with the length of the segments being removed. The sum of these lengths is

$$\frac{1}{4} + \frac{2}{16} + \frac{4}{64} + \frac{8}{256} + \cdots + \frac{2^n}{4^{n+1}} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{4^{n+1}} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{4}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2}.$$

Then the measure of F is $1 - \frac{1}{2} = \frac{1}{2}$.

3. We do the same but remove the middle segments of length $(\frac{1}{5})^n$ at each stage. Then the measure of the resulting Cantor set is

$$1 - \sum_{n=0}^{\infty} \frac{2^n}{5^{n+1}} = \cdots = \frac{2}{5},$$ as you will check.

4. Let $E_0 = [0,1]$. Let $E_1 = E_0$ with the interior of the middle $1/3!$ segment removed. Let $E_2 = E_1$ with the interiors of the two middle segments on length $1/4!$. Etc. Etc.
Let $E = \bigcap_{n=0}^{\infty} E_n$. Then the measure of E is

$$1 - \sum_{n=0}^{\infty} \frac{2^n}{(n+3)!} \approx 0.7013679876 4$$

as you will check. Hint: Consider the Taylor series of e^x at $x=2$.

Remark The concept of "measure" is very important and will come up again. It is the central idea in Math 501. Here we are being very informal. But think about this: what is the "length" or measure of $[0,1] \cap \mathbb{Q}$? What about $[0,1] - \mathbb{Q}$?

The textbook calls Cantor sets in $\mathbb{R}$ with nonzero measure fat Cantor sets, but perhaps a better term is thick Cantor sets. But terms are used in the published literature.

Here are three papers and a book about thick Cantor sets.

Williams, R. F. How big is the intersection of two thick Cantor sets? *Continuum theory and dynamical systems (Arcata, CA, 1989)*, 163–175, Contemp. Math., 117, *Amer. Math. Soc., Providence, RI*, 1991.

Kraft, Roger Intersections of thick Cantor sets. *Mem. Amer. Math. Soc.* 97 (1992), no. 468, vi+119 pp.

Hunt, Brian R.; Kan, Ittai; Yorke, James A. When Cantor sets intersect thickly. *Trans. Amer. Math. Soc.* 339 (1993), no. 2, 869–888.

Clark, Tyler; Richmond, Tom Cantor sets arising from continued radicals. *Ramanujan J.* 33 (2014), no. 3, 315–327.

But, who cares?

It turns out Cantor sets are very useful in several areas of mathematics, including chaos theory and information theory. Here we give 3 examples to illustrate this.

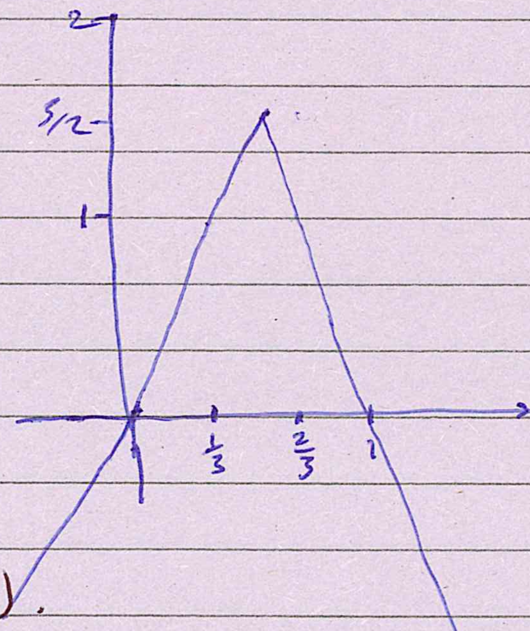
(1) Dynamics of the tent map.

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 3x & x \le \frac{1}{2} \\ 3-3x & x \ge \frac{1}{2} \end{cases}$.

See the graph. We want to iterate f, this is to apply it over and over. Let $x_0 \in \mathbb{R}$.

Let $x_1 = f(x_0)$,
$x_2 = f(x_1)$
$x_3 = f(x_2)$, etc.

The sequence $(x_0, x_1, x_2, \ldots)$ is called the orbit of x_0



The orbit of 0 is $(0, 0, 0, \ldots)$.
The orbit of $\frac{1}{3}$ is $(\frac{1}{3}, 1, 0, 0 \ldots)$. The orbit of 2 is $(2, -3, -9, -27, \ldots -(3)^n \ldots)$. The orbit of $\frac{1}{2}$ is $(\frac{1}{2}, -\frac{3}{2}, -\frac{9}{2}, -\frac{27}{2}, \ldots \quad)$. We say the orbits of 2 and $1/2$ diverge to $-\infty$. In fact all the values in $(-\infty, 0) \cup (\frac{1}{3}, \frac{2}{3}) \cup (1, \infty)$ diverge to $-\infty$.

What about values in $(\frac{1}{9}, \frac{2}{9})$. They will get mapped to $(\frac{1}{3}, \frac{2}{3})$ and hence will diverge to $-\infty$. The same is true for $(\frac{7}{9}, \frac{8}{9})$.

Points that do not div. to $-\infty$ stay within $[0,1]$ for all iterations of f. This set is called the invariant set of f.

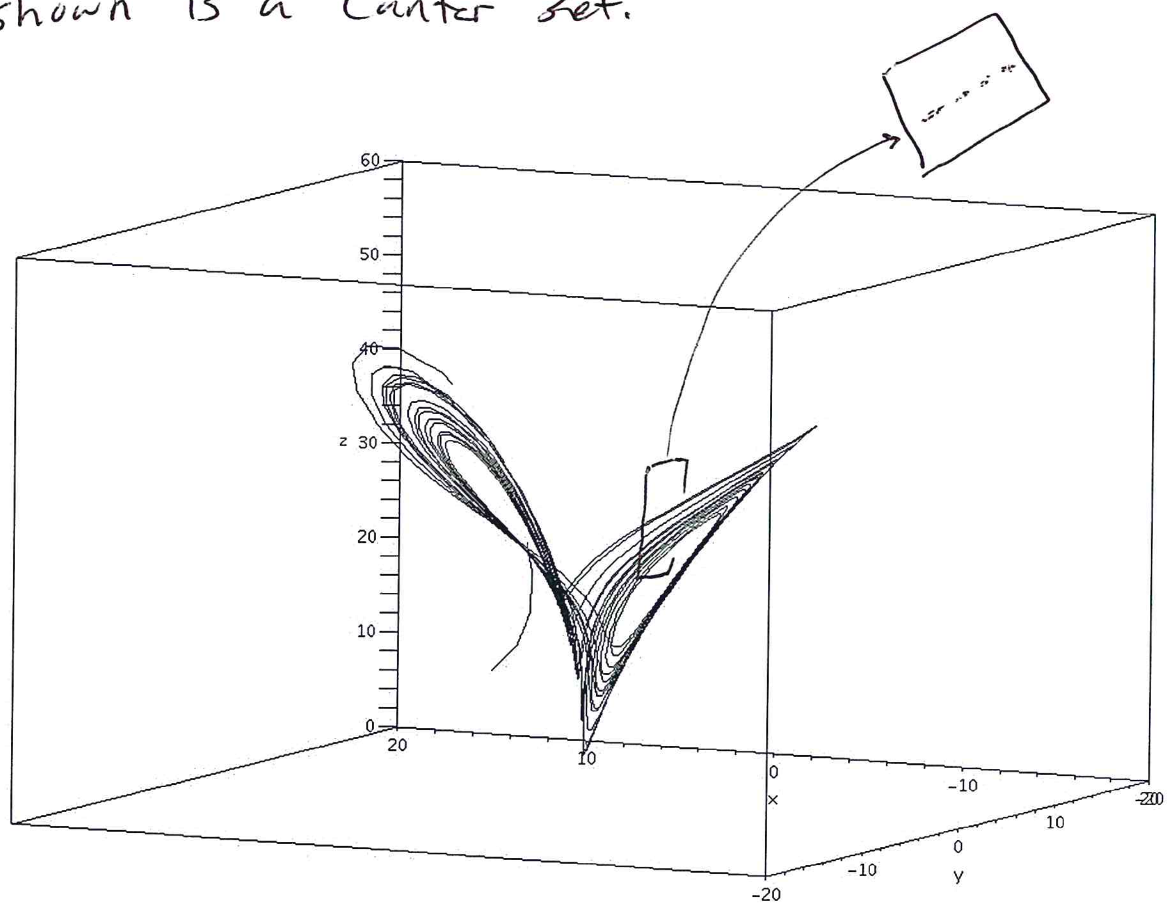
The invariant set for f is the middle thirds Cantor set C. You should check this!

Find a function whose invariant set is the middle fourths Cantor set.

Find a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is $C \times C$.

(2) Lorenz Attractor

You may have heard of the Lorenz Equations. They are a 3×3 system of nonlinear differential equations that opened up the field of chaos theory. Below I have plotted a solution curve with Maple that moves quickly toward the "strange attractor." The cross section shown is a Cantor set.



Look up the papers of R.F. Williams.

3

Symbolic Dynamics

Let $\Sigma = \{ (s_i)_{i=-\infty}^{\infty} \mid s_i = 0 \text{ or } 1 \}$. For $a, b \in \Sigma$ define

$$d(a,b) = \sum_{i=-\infty}^{\infty} \frac{|a_i - b_i|}{2^{|i|}}.$$

Then d is a metric for Σ and (Σ, d) is a Cantor space. This example, and many variations of it, are ubiquitous in the fields symbolic dynamics and information theory.

See: "An Introduction to Symbolic Dynamics and Coding," by Douglas Lind and Brian Marcus, Cambridge University Press, 1995.