Ch 2
Sec 10

Completion

Thm Every metric space can be completed. This means $\exists$ a complete m. sp. $\hat{M}$ and $h: M \to \hat{M}$ s.t. $\overline{h(M)} = \hat{M}$ and $h: M \to h(M)$ is an isometry. (An isometry is a homeomorphism that preserves distances.)

Idea Take the collection of Cauchy sequences in M and declare two to be equivalent if $d(p_n, q_n) \to 0$. We can put a "natural" metric on the set of equivalence classes to get $\hat{M}$.

Pf Let $\bar{p} = (p_n)$ and $\bar{q} = (q_n)$ be Cauchy seq's in M. Define $\bar{p} \sim \bar{q}$ if $d(p_n, q_n) \to 0$. It is easy to check that $\sim$ is an eq. rel. (Do This!) Let $\hat{M} = \{$ Cauch seq's in $M \}$ mod $\sim$.

We define a metric D on $\hat{M}$ by letting

$$D(P, Q) = \lim_{n \to \infty} d(p_n, q_n)$$

where $(p_n) \in P$ and $(q_n) \in Q$. We need to check the following.

(a) D is a well defined function, meaning that $D(P,Q)$ always exists and does not depend on which seq's are choosen from P and Q.

(b) D is a metric

(c) $\hat{M}$ is complete.

Let $h: M \to \hat{M}$ be given by $h(p) = [(p,p,p,p,\dots)]$. We must show the following.

(d) h is an isometry

(e) $\overline{h(M)} = \hat{M}$, i.e. $h(M)$ is dense in $\hat{M}$.

Pf of (a) We use Lemma 81 (pg 119) from the textbook:

$$\forall p,q,x,y \in M, \ |d(p,q) - d(x,y)| \leq d(p,x) + d(q,y).$$

Let $\bar{p}, \bar{p}'$ be seq's in P and $\bar{q}, \bar{q}'$ be seq's in Q. By Lemma 81

$$|d(p_m, q_m) - d(p_n, q_n)| \leq d(p_m, p_n) + d(q_m, q_n).$$

Therefore $(d(p_n, q_n))$ is a Cauchy seq in $\mathbb{R}$ and hence $\lim_{n\to\infty} (p_n, q_n)$ exists.

Let $L = \lim d(p_n, q_n)$ and $L' = \lim d(p_n', q_n')$.
We claim $L = L'$. This is just the triangle ineq.

$$|L - L'| \le |L - d(p_n, q_n)| + |d(p_n, q_n) - d(p_n', q_n')| + |d(p_n', q_n') - L'|$$

The first and third term on the RHS go to zero as $n \to \infty$. The middle term, by Lemma 81, is

$$\le d(p_n, p_n') + d(q_n, q_n').$$

These two terms go to zero also. Thus $L = L'$, and hence D is independent of the choice of sequences used from the eq. classes P and Q.

Pf of (b) (D is a metric.)

Since the original metric ~~and~~ d on M is symmetric, That is,

$$D(P, Q) = \lim d(p_n, q_n) = \lim d(q_n, p_n) = D(Q, P).$$

Clearly,

$$D(P, P) = \lim d(p_n, p_n) = \lim 0 = 0.$$

If $D(P, Q) = 0$, then for $(p_n) \in P$ and $(q_n) \in Q$ $\lim d(p_n, q_n) = 0$. But then $(p_n) \sim (q_n)$ and so $P = Q$. This shows positive definitness.

Finally, let $P, Q, R \in \hat{M}$. We have the ~~[illegible]~~ tri. ineq:

$$D(P,Q) = \lim d(p_n, q_n) \leq \lim d(p_n, r_n) + d(r_n, q_n)$$

$$= \lim d(p_n, r_n) + \lim d(r_n, q_n) = D(P,R) + D(R,Q).$$

Thus D is a metric on $\hat{M}$.

Pf of ⓒ ($\hat{M}$ is complete.) Let (P_k) be a Cauchy seq in $\hat{M}$. We must show it converges in $\hat{M}$. We will construct a candidate for the limit using a diagonalization technique. To do this we will carefully select $(p_n^k) \in P_k$ for each k.

First selection any $(p_n^k) \in P_k$. $\exists N$ s.t. $m, n \geq N \Rightarrow d(p_m^k, p_n^k) < \frac{1}{k}$. Consider the ~~[illegible]~~ subseq obtained by deleting the first N terms: $(p_{n+N}^k)_{n=1}^{\infty}$. It is in P_k. In fact any subseq of a Cauchy seq will be Cauchy and will be equivalent in our sense to the original sequence. ~~Proof~~ Prove this!

Renumber the seq's $(p_n^k)_{n=N+1}^{\infty}$ so that they each start from p_1^k. Thus $\forall k$ $(p_n^k)_{n=1}^{\infty} \in P_k$ and $d(p_a^k, p_b^k) < \frac{1}{k}$ $\forall\, a,b \geq 1$.

Let $q_n = p_n^n$. Let $Q = [(q_n)]$. We need to check that (q_n) is Cauchy, so that $Q \in \hat{M}$, and that $P_k \to Q$ in $\hat{M}$.

<u>(q_n) is Cauchy</u>: Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $k, \ell \geq N \Rightarrow D(P_k, P_\ell) < \frac{\varepsilon}{3}$. Further, we can choose $N > 3/\varepsilon$. Thus $1/k$ and $1/\ell$ are $< \varepsilon/3$. Now,

$$d(q_k, q_\ell) = d(p_k^k, p_\ell^\ell) \leq d(p_k^k, p_n^k) + d(p_n^k, p_n^\ell) + d(p_n^\ell, p_\ell^\ell)$$

$$< \frac{1}{k} + d(p_n^k, p_n^\ell) + \frac{1}{\ell} < d(p_n^k, p_n^\ell) + \frac{2\varepsilon}{3}. \quad (*)$$

The $\lim\limits_{n\to\infty} d(p_n^k, p_n^\ell) = D(P_k, P_\ell) < \varepsilon/3$. Thus for n large enough $d(p_n^k, p_n^\ell) < \varepsilon/3$. Since the inequality $(*)$ is valid $\forall n \geq 1$ we have

$$d(q_k, q_\ell) < \varepsilon.$$

Thus, (q_n) is Cauchy and $Q \in \hat{M}$.

$P_k \to Q$

Let $\varepsilon > 0$. Choose $N > \frac{2}{\varepsilon}$ s.t. $k, n \geq N \Rightarrow d(q_k, q_n) < \frac{\varepsilon}{2}$.
Then

$$d(p_n^k, q_n) \leq d(p_n^k, p_k^k) + d(p_k^k, q_n)$$

$$= d(p_n^k, p_k^k) + d(q_k, q_n)$$

$$< \frac{1}{k} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now $\lim_{n\to\infty} d(p_n^k, q_n) = D(P_k, Q)$. We have that for large enough k, $D(P_k, Q) < \varepsilon$.

Thus $\lim_{k\to\infty} P_k = Q$ as required.

Pf of (d) Recall $h: M \to \hat{M}$ was given by $h(p) = [(p, p, p, \ldots)]$.
If $p \neq q$, then $D((p), (q)) = |p - q| \neq 0$. Thus $h(p) \neq h(q)$, so h is one-to-one. This also shows that h preserves distances.
This fact can be used to show that h, and h^{-1} on $h(M)$, are continuous. Just use $\varepsilon = \delta$.

Pf of (e) We need to show $h(M)$ is dense in $\hat{M}$. But by construction if $Q \in \hat{M}$, $\exists\, P_k \in h(M)$, $k = 1, 2, 3 \ldots$ s.t. $P_k \to Q$.

$\boxed{*}$