Subspaces and Inheritance

Def Let (M, d) be a metric space. Let $N \subset M$. Then d restricted to $N \times N$ gives a metric for N and so (N, d) is called a subspace of M.

Q? Suppose $U \subset N \subset M$ and U is open in N. Does it follow that U is open in M?

No. For example, let $N = [0,1] \subset \mathbb{R}$ be regarded as a subspace. Then $[0, \frac{1}{3}) =$ the open ball in N with center 0 and radius $1/3$. Thus $[0, 1/3)$ is an open subset of N, but it is not open in M.

The same issue arises with closed subsets. Let $P = (0, 1) \subset \mathbb{R}$. Then $(0, \frac{1}{2}]$ contains all its limit points in P and thus is closed in P. But $(0, \frac{1}{2}]$ is not closed in $\mathbb{R}$.

Thus, openness and closedness become $\boxed{\text{relative}}$ concepts.

However we can make the follow statements.

Thm Let $N \subset M$, where M is a metric space and we use the subspace metric for N. Then $U \subset N$ is open iff $\exists$ an open set $V \subset M$ s.t.

$$U = N \cap V.$$

Likewise, $C \subset N$ is closed iff $\exists$ a closed set $K \subset M$ s.t.

$$C = M \cap K.$$

Proof I'll proof the first claim and leave the second to you. Suppose $U \subset N$ is open in N. Then

$$U = \bigcup_{p \in U} B_N(p, \varepsilon_p)$$

where each $\varepsilon_p > 0$ is choosen so that $B_N(p, \varepsilon_p) \subset N$. Now any open ball in N is the intersection of an open ball in M with N. Thus $\forall p \in U$

$$B_N(p, \varepsilon_p) = B_M(p, \varepsilon_p) \cap N.$$

Let $V = \bigcup_{p \in U} B_M(p, \varepsilon_p)$. Then $U = N \cap V$.

For the other direction, let $V \subset M$ be open and let $U = N \cap V$. Let $p \in U$. $\exists\, \varepsilon > 0$ s.t.

$$B_M(p, \varepsilon) \subset V$$

Since V is open. But $B_N(p,\varepsilon) = N \cap B_M(p,\varepsilon) \subset U$. Thus U is open in the subspace N. $\blacksquare$

The following facts are easy to prove

If N is open in M, then $U \subset N$ is open in N iff U is open in M.

If N is closed in M, then $C \subset N$ is closed in N iff C is closed in M.