

Ch 4
Sec 1

Function Spaces

Uniform vs Pointwise convergence.

Def For each $n \in \mathbb{N}$ let $f_n: [a, b] \rightarrow \mathbb{R}$, and $f: [a, b] \rightarrow \mathbb{R}$.

If for each $x \in [a, b]$ the $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ then

we say f is the pointwise limit of (f_n) .

If $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

$\forall x \in [a, b]$, then we say f_n converges uniformly to f on $[a, b]$.

Notation

$f_n \rightarrow f$ means pointwise convergence.

$f_n \rightrightarrows f$ means uniform convergence.

Ex

Let $f_n: [0, 1] \rightarrow \mathbb{R}$ be given by $f_n(x) = x^n$.

Let $f_n: [0, 1] \rightarrow \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1. \end{cases}$$

Then $f_n \rightarrow f$ but $f_n \not\rightrightarrows f$.

You should check this.

Thm If $f_n \Rightarrow f$ and each f_n is continuous at x_0 then f is continuous at x_0 also.

Pf Let $f_n: [a, b] \rightarrow \mathbb{R}$ be continuous at $x_0 \in [a, b]$.
Assume $f_n \Rightarrow f$ on $[a, b]$.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \forall x \in [a, b].$$

$\exists \delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

Thus, when $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon$$



Def Let $C_b = C_b([a, b], \mathbb{R}) =$ all bdd functions $[a, b] \rightarrow \mathbb{R}$.

The sup norm on C_b is given by

$$\|f\| = \sup \{|f(x)| : x \in [a, b]\}.$$

You can check that norm criteria are satisfied:

$$\|f\| \geq 0, \|f\| = 0 \text{ iff } f(x) = 0, \forall x.$$

$$\|cf\| = |c| \|f\|$$

$$\|f+g\| \leq \|f\| + \|g\|.$$

This gives us a metric on C_b ,

$$d(f, g) = \sup \{|f(x) - g(x)|\} = \|f - g\|.$$

Thm Convergence w.r.t. the sup metric is equivalent to uniform convergence.

Pf Easy. See textbook, Thm 2, pg 216.

Thm C_b with the sup metric is a complete metric sp.

Pf Let (f_n) be a Cauchy seq in C_b . First, we will show that it has a pointwise limit. Then we will show that the convergence is uniform. Finally, we will show that this limit is bdd, and hence in C_b .

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be s.t. Let $x_0 \in [a, b]$.

$$m, n \geq N \Rightarrow d(f_n, f_m) < \epsilon.$$

\hookrightarrow sup metric.

Thus,

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x \in [a, b]} \{|f_n(x) - f_m(x)|\} = d(f_n, f_m) < \epsilon.$$

Therefore, $\lim_{n \rightarrow \infty} f_n(x_0)$ exist for any $x_0 \in [a, b]$.

Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for each $x \in [a, b]$.

Now we show that the convergence is in fact uniform. Let $\epsilon > 0$. Let N_1 be s.t. $m, n \geq N_1$ implies

$$d(f_m, f_n) < \epsilon/2.$$

Let $x \in [a, b]$. Let N_2 be s.t. $n \geq N_2$ implies

$$|f_n(x) - f(x)| < \epsilon/2.$$

Let $n \geq N_1$ and $n_2 \geq \max\{N_1, N_2\}$. Then

$$|f_{n_1}(x) - f(x)| \leq \underbrace{|f_{n_1}(x) - f_{n_2}(x)|}_{\uparrow n_1} + \underbrace{|f_{n_2}(x) - f(x)|}_{\uparrow n_2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

Since N_1 is independent of x , the convergence is uniform.

Lastly, we need to show f is bdd. $\exists N \in \mathbb{N}$ s.t. $\|f_N - f\| < 1$. Thus,

$$|f_N(x) - f(x)| < 1 \quad \forall x \in [a, b].$$

Since $f_N \in C_b \exists M \geq 0$ s.t. $|f_N(x)| < M \quad \forall x \in [a, b]$.
But now we know that $|f(x)| < M + 1$. Thus $f \in C_b$.



Corollary $C^0([a, b], \mathbb{R})$ is a closed, complete subspace of $C_b([a, b], \mathbb{R})$.

Pf Two lines. Figure it out.

Next we look at series of functions: $\sum_{k=0}^{\infty} f_k(x)$.

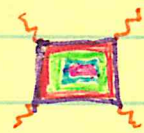
Thm (The Weierstrass M-test) For $k=0, 1, 2, 3, \dots$
let $f_k(x) \in C_b$ and suppose $\|f_k\| \leq M_k$. If
 $\sum M_k$ convergence, then $\sum f_k(x)$ converges uniformly.

Pf Let $F_n(x) = \sum_{k=0}^n f_k(x)$, $\forall x \in [a, b]$. Let $n > m$.
Then

$$\begin{aligned} d(F_n, F_m) &\leq d(F_n, F_{n-1}) + d(F_{n-1}, F_{n-2}) + \dots + d(F_{m+1}, F_m) \\ &= \|F_n - F_{n-1}\| + \dots + \|F_{m+1} - F_m\| = \|f_n\| + \dots + \|f_{m+1}\| \leq \sum_{k=m+1}^n M_k. \end{aligned}$$

Let $\epsilon > 0$. Then $\exists N \geq 0$ s.t. $n > m \geq N \Rightarrow \sum_{k=m+1}^n M_k < \epsilon$.

Thus $n, m \geq N \Rightarrow d(F_n, F_m) < \epsilon$. Thus $(F_n) = (\sum_{k=0}^n f_k(x))$
converges uniformly.



Now we present some basic results involving integrability, and differentiability.

Thm

Let $f_n: [a, b] \rightarrow \mathbb{R}$ be R.I. (hence $f_n \in C_b$).

Suppose $f_n \Rightarrow f$. Then

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx.$$

pf

By the RLT each $f_n \in C_b$. Let Z_n be the set of points in $[a, b]$ where f_n is not continuous. By the RLT Z_n is a zero set. Then $Z = \cup Z_n$ is a zero set. Each f_n is cont. $\forall x \in [a, b] - Z$. Thus, f is cont on $[a, b] - Z$. We know that $f \in C_b$. Thus $f \in \mathcal{R}$.

To finish, we have

$$\begin{aligned} \left| \int_a^b f(x) dx - \int_a^b f_n(x) dx \right| &= \left| \int_a^b f(x) - f_n(x) dx \right| \\ &\leq \int_a^b |f(x) - f_n(x)| dx \leq (b-a) d(f, f_n) \rightarrow 0. \end{aligned}$$

$$\text{Thus, } \int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx.$$



Cor

If $f_n \in \mathcal{R}$, $n=1,2,3,\dots$ and $f_n \Rightarrow f$, then

$$\int_a^x f_n(t) dt \Rightarrow \int_a^x f(t) dt.$$

Cor

If $\sum_{k=0}^{\infty} f_k \Rightarrow F$, each $f_k \in \mathcal{R}$, then

$$\sum_{k=0}^{\infty} \int_a^b f_k(x) dx = \int_a^b \sum_{k=0}^{\infty} f_k(x) dx.$$

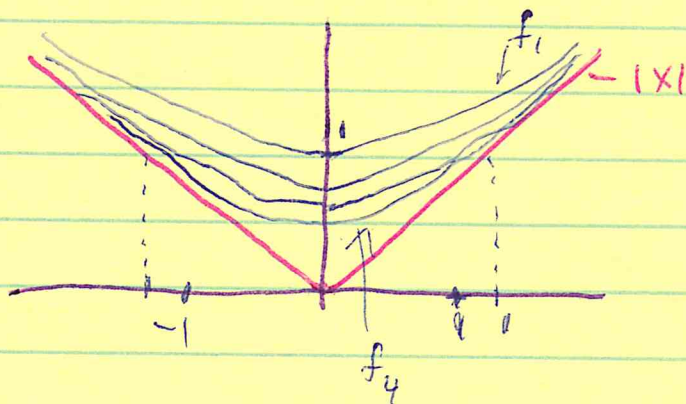
See textbook for proofs of the corollaries,
~~Corollary 7~~ Corollary 7, pg 218 and Thm 8 pg 219.

The situation for differentiability is different

Ex

Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ for $x \in [-1, 1]$. Each f_n is differentiable. The $\lim_{n \rightarrow \infty} f_n(x) = \sqrt{x^2} = |x|$,

which is not diff. at $x=0$. You can check that the convergence is uniform.



But, we do have the following.

Thm Suppose $f_n \rightrightarrows f$ and $f'_n \rightrightarrows g$. Then $f' = g$.

Pf Let

Pick $x \in [a, b]$.

$$\phi_n(t) = \begin{cases} \frac{f_n(t) - f_n(x)}{t - x} & t \neq x \\ f'_n(x) & t = x, \end{cases}$$

$$\phi(t) = \begin{cases} \frac{f(t) - f(x)}{t - x} & t \neq x \\ g(x) & t = x. \end{cases}$$

Each $\phi_n(t)$ is cont. (Check limit as $t \rightarrow x$.)

Clearly $\phi_n(x) \rightarrow \phi(x)$, pointwise. We claim $\phi_n \rightrightarrows \phi$.

$\forall m, n$ the MVT gives

$$\phi_m(t) - \phi_n(t) = \frac{[f_m(t) - f_n(t)] - [f_m(x) - f_n(x)]}{t - x} = f'_m(\theta) - f'_n(\theta)$$

for some θ between t and x . Since $f'_n \rightrightarrows g$, $f'_m - f'_n \rightrightarrows g - g = 0$ as $m, n \rightarrow \infty$, (This means $\forall \epsilon > 0 \exists N$ s.t. $m, n \geq N \rightarrow d(f'_m, f'_n) < \epsilon$.) it follows that

The seq. (ϕ_n) is Cauchy in C^0 . Since C^0 is complete $\exists \psi \in C^0$ s.t. $\phi_n \rightrightarrows \psi$.

Since $\phi_n(x) \rightarrow \phi(x)$, pointwise, we know $\psi(x) = \phi(x)$.

Since ψ is cont (check $\lim_{t \rightarrow x} \psi(t) = \psi(x)$), we have

$f'(x) = g(x)$ as desired.



of diff. function.

Cor

Let $\sum f_k(x)$ be a uniformly convergent series, and suppose $\sum f'_k(x)$ also converges uniformly, then

$$\left(\sum_{k=0}^{\infty} f_k(x) \right)' = \sum_{k=0}^{\infty} f'_k(x)$$

Pf

See textbook, (Thm 10, pg 220).