

Ch 4
Sec 4

Uniform Approximation in C^0

Thm Weierstrass Approximation Thm.

The set of polynomials is dense in $C^0([a, b], \mathbb{R})$.

The textbook gives two proofs. I'll cover the first and you should read/study the second on your own.

The first proof uses Bernstein polynomials. They are also used in computer graphics to create Bézier curves. We will restrict our attention to the interval $[0, 1]$ for simplicity.

Def For each $n \in \mathbb{N} \cup \{0\}$ there are $n+1$ Bernstein basis polynomials of degree n . They are

$$b_{k,n} = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } k=0, 1, 2, \dots, n,$$

where,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Next we list a few.

$$\begin{aligned} b_{00}(x) &= 1 \\ b_{01}(x) &= (1-x), \quad b_{11}(x) = x \\ b_{02}(x) &= (1-x)^2, \quad b_{12}(x) = 2x(1-x), \quad b_{22}(x) = x^2 \\ b_{03}(x) &= (1-x)^3, \quad b_{13}(x) = 3x(1-x)^2, \quad b_{23}(x) = 3x^2(1-x), \quad b_{33}(x) = x^3 \\ &\text{etc.} \end{aligned}$$

Then any polynomial of the form

$$\sum_{k=0}^n c_k b_{kn}(x),$$

is called a Bernstein poly. of degree n .

Facts

We state and prove two handy facts about Bernstein poly's.

$$\sum_{k=0}^n b_{kn}(x) = 1. \quad (\star)$$

Thus for each n the set $\{b_{kn}\}$ forms what is called a "partition of unity". The proof follows from the binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}, \quad (\dagger)$$

by letting $y = 1-x$.

The other identity we will need is

$$\sum_{k=0}^n (k-nx)^2 b_{kn}(x) = nx(1-x). \quad (\#)$$

The proof of this is a little harder. It goes like this. Apply $\frac{d}{dx}$ to both sides of (6), twice. Thus,

$$n(x+y)^{n-1} = \sum_{k=0}^n \binom{n}{k} k x^{k-1} y^{n-k} \quad \text{and}$$

$$n(n-1)(x+y)^{n-2} = \sum_{k=0}^n \binom{n}{k} k(k-1) x^{k-2} y^{n-k}.$$

Now let $y=1-x$ in both of these equations. Thus,

$$n = \sum_{k=0}^n \binom{n}{k} k x^{k-1} (1-x)^{n-k} \quad \text{and}$$

$$n(n-1) = \sum_{k=0}^n \binom{n}{k} k(k-1) x^{k-2} (1-x)^{n-k}.$$

Multiply the first by x and the second by x^2 . Thus,

$$nx = \sum_{k=0}^n \binom{n}{k} k x^k (1-x)^{n-k} = \sum_{k=0}^n k b_{kn}(x) \quad \text{and}$$

$$n(n-1)x^2 = \sum_{k=0}^n \binom{n}{k} k(k-1) x^k (1-x)^{n-k} = \sum_{k=0}^n k(k-1) b_{kn}(x)$$

$$= \sum_{k=0}^n k^2 b_{kn}(x) - \underbrace{\sum_{k=0}^n k b_{kn}(x)}_{= nx}$$

Hence,

$$\sum_{k=0}^n k^2 b_{kn}(x) = n(n-1)x^2 + nx.$$

Now we go back to (#):

$$\sum_{k=0}^n (k-nx)^2 b_{kn}(x) = \sum_{k=0}^n k^2 b_{kn}(x) - 2nx \sum_{k=0}^n k b_{kn}(x) + n^2 x^2 \sum_{k=0}^n b_{kn}(x)$$

$$= n(n-1)x^2 + nx - 2nx(nx) + n^2 x^2$$

$$= \underline{n^2 x^2} - \underline{nx^2} + nx - \underline{2n^2 x^2} + \underline{n^2 x^2}$$

$$= nx(1-x).$$

Thus (#) is proved.

pf

(Proof of the Weierstrass Approx. Thm)

Let $f \in C^0([0,1], \mathbb{R})$. For each $n \in \mathbb{N}$ define

$$P_n(x) = \sum_{k=0}^n \binom{n}{k} c_k x^k (1-x)^{n-k} = \sum_{k=0}^n c_k b_{kn}(x)$$

where $c_k = f(\frac{k}{n})$. (Of course c_k depends on n and k .)

$$\text{Note, } f(x) = f(x) \cdot 1 = f(x) \sum_{k=0}^n b_{kn}(x) = \sum_{k=0}^n f(x) b_{kn}(x).$$

We will show $P_n \rightrightarrows f$. Notice that as $n \rightarrow \infty$ the partition $\{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}\}$ gets finer.

Fix a value of $x \in [0,1]$ for now. Let $\epsilon > 0$.

Since $f(x)$ is uniformly cont. on $[0,1]$, $\exists \delta > 0$, s.t.

$$|t-s| < \delta \Rightarrow |f(t) - f(s)| < \frac{\epsilon}{2}.$$

Define

$$K_n = \{k \in \{0, 1, 2, \dots, n\} \mid |\frac{k}{n} - x| < \delta\} \text{ and}$$

$$K'_n = \{0, 1, 2, \dots, n\} - K_n.$$

Now,

$$|P_n(x) - f(x)| = \left| \sum_{k=0}^n (c_k - f(x)) b_{kn}(x) \right| \leq \sum_{k=0}^n |c_k - f(x)| b_{kn}(x)$$

$$= \sum_{k \in K_n} |c_k - f(x)| b_{kn}(x) + \sum_{k \in K'_n} |c_k - f(x)| b_{kn}(x).$$

Since $c_k = f(\frac{k}{n})$, $|c_k - f(x)| < \frac{\epsilon}{2}$ for $k \in K_n$.

$$\text{Thus, } \sum_{k \in K_n} |c_k - f(x)| b_{kn}(x) < \frac{\epsilon}{2} \sum_{k \in K_n} b_{kn}(x) \leq \frac{\epsilon}{2}$$

For $k \in K'_n$ we have

$$|\frac{k}{n} - x| \geq \delta \Rightarrow |k - nx|^2 \geq (n\delta)^2$$

Now by (#) we have

$$\begin{aligned} n x(1-x) &= \sum_{k=0}^n (k-nx)^2 b_{kn}(x) \geq \sum_{k \in K'_n} (k-nx)^2 b_{kn}(x) \\ &\geq \sum_{k \in K'_n} (n\delta)^2 b_{kn}(x). \end{aligned}$$

Therefore,

$$\sum_{k \in K'_n} b_{kn}(x) \leq \frac{n x(1-x)}{(n\delta)^2} \leq \frac{1}{4n\delta^2}$$

Since the max of $x(1-x)$ over $[0,1]$ is $\frac{1}{4}$.

Let $M = \|f\|$. Then $\|c_k - f(x)\| = |f(\frac{k}{n}) - f(x)| \leq 2M$.

Thus,

$$\sum_{k \in K'_n} |c_k - f(x)| b_{kn}(x) \leq \frac{M}{2n\delta^2} < \frac{\epsilon}{2}$$

for n large enough.

Thus, $\exists N$ s.t. $n \geq N \Rightarrow |p_n(x) - f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

But for uniform convergence N must be independent of our choice of $x \in [0, 1]$. The sets K_n, K_n' clearly did depend on x . But, the bounds on the two sums, \bullet and \bullet , did not. Thus the convergence is uniform.



Ex

Let $f: [0, 1] \rightarrow \mathbb{R}$ be given by $f(x) = |x - \frac{1}{2}|$. Find the smallest n so that $\|p_n - f\| < 0.1$.

On the next page is a Maple worksheet. Letting $n=15$ gives $\|p_n - f\| = 0.1047\dots$, while $n=16$ gives $\|p_n - f\| = 0.09819\dots$. Thus, the smallest value for n is 16.

Repeat for $f(x) = ||x - \frac{1}{2}| - \frac{1}{4}|$ and $\|p_n - f\| < 0.02$. This time we get $n=398$.

Bernstein Polynomial Approximations

Find the first value of n so the the n^{th} Bernstein polynomial for $f(x) = |x-0.5|$ has $\|p_n - f\| < 0.1$.

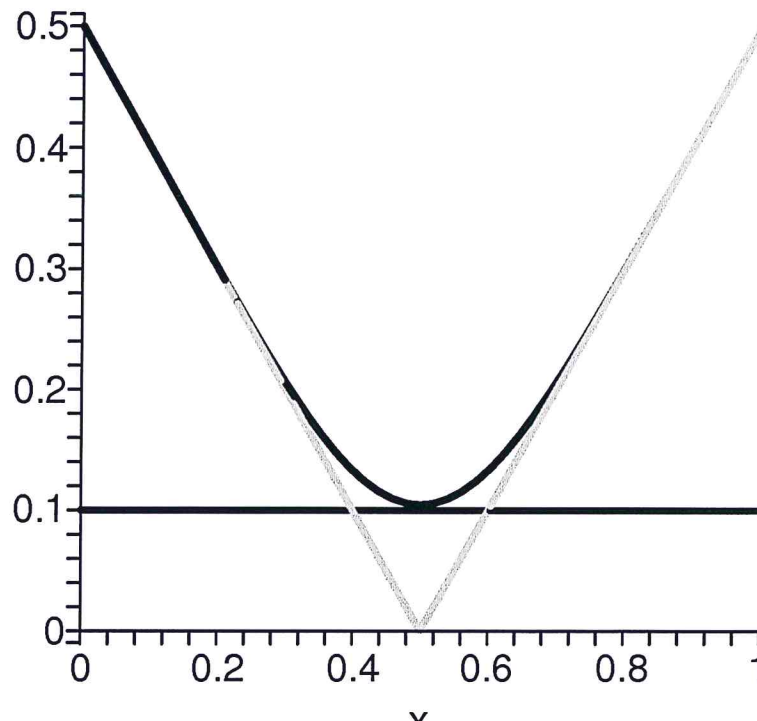
```
> pn := x -> sum((n!/k!/(n-k)!)*abs(k/n-0.5)*x^k*(1-x)^(n-k), k=0..n);
```

$$pn := x \rightarrow \sum_{k=0}^n \frac{n! \left| \frac{k}{n} - 0.5 \right| x^k (1-x)^{(n-k)}}{k!(n-k)!}$$

```
> n:=15;
```

$n := 15$

```
> plot([0.1,abs(x-0.5),pn(x)],x=0..1,color=[black,gray,black],
linestyle=[2,1,1],thickness=2);
```



```
> pn(0.5);
```

0.1047363281

```
> n:=16;
```

$n := 16$

```
> pn(0.5);
```

0.09819030762

The answer is n=16.

Repeat for $f(x) = ||x-0.5| - 0.25|$ and $||p_n - f| < 0.02$.

```
> f:=x-> abs(abs(x-0.5)-0.25);
```

$f := x \rightarrow ||x - 0.5| - 0.25|$

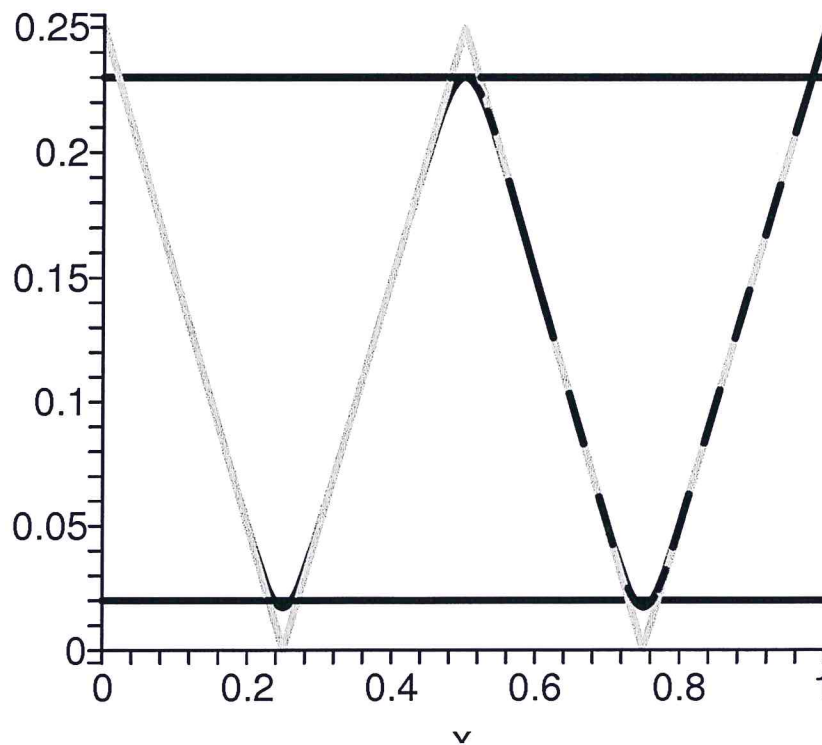
```
> pn:= x-> sum((n!/k!/(n-k)!)*f(k/n)*x^k*(1-x)^(n-k),k=0..n);
```

$$pn := x \rightarrow \sum_{k=0}^n \frac{n! f\left(\frac{k}{n}\right) x^k (1-x)^{(n-k)}}{k!(n-k)!}$$

```
> n:=398;
```

n := 398

```
> plot([0.02,0.23,f(x),pn(x)],x=0..1,color=[black,black,gray,black],  
linestyle=[2,2,1,1],thickness=2);
```



```
> pn(0.5);
```

0.2300153875

The answer is n=398.

The Stone-Weierstrass Thm.

Defs

Let M be a compact metric space. (It follows that M is complete.) Let $\mathcal{A} \subset C^0(M, \mathbb{R})$. Then \mathcal{A} is called a **function algebra** if $\forall f, g \in \mathcal{A}, c \in \mathbb{R}$ we have $f+g, cf, f \cdot g$ are in \mathcal{A} .

A function algebra **vanishes at $p \in M$** if $\forall f \in \mathcal{A}, f(p) = 0$.

A func. alg. **separates points** if $\forall p_1, p_2 \in M, p_1 \neq p_2, \exists f \in \mathcal{A}$ s.t. $f(p_1) \neq f(p_2)$.

Ex

Polynomials in $C^0([a, b], \mathbb{R})$ form a nonvanishing func. alg that sep. pts.

Ex

$\{ p(x)(x-3) \mid p(x) \text{ any poly} \}$ vanishes at $p=3$.

Ex

$\{ p(x)(x^2-1)+2 \mid p(x) \text{ any poly} \}$ does not sep. 1 and -1.

Thm

(Stone-Weierstrass) Let M be a compact m. sp. and $\mathcal{A} \subset C^0(M, \mathbb{R})$ a func. alg. that vanishes nowhere and sep. pts. Then \mathcal{A} is dense in $C^0(M, \mathbb{R})$.

This is a generalization of the W. Approx Thm.
We need some lemmas.

Lemma

If \mathcal{A} vanishes nowhere and sep. pts, then
 $\forall p_1, p_2 \in M, p_1 \neq p_2$ and $c_1, c_2 \in \mathbb{R}, \exists f \in \mathcal{A}$ s.t.

$$f(p_1) = c_1 \text{ and } f(p_2) = c_2.$$

pf

$\exists g_1, g_2 \in \mathcal{A}$ s.t. $g_1(p_1) \neq 0$ and $g_2(p_2) \neq 0$.

Let $g(x) = [g_1(x)]^2 + [g_2(x)]^2$. Then $g \in \mathcal{A}$ and
 $g(p_1) \neq 0, g(p_2) \neq 0$. $\exists h \in \mathcal{A}$ s.t. $h(p_1) \neq h(p_2)$.

Consider the matrix $H = \begin{bmatrix} g(p_1) & g(p_1)h(p_1) \\ g(p_2) & g(p_2)h(p_2) \end{bmatrix} = \begin{bmatrix} a & ab \\ c & cd \end{bmatrix}$.

Now $a \neq 0, c \neq 0$ and $b \neq d$. Thus $\det H =$
 $acd - abc = ac(b-d) \neq 0$. Therefore, $\exists!$ solution to

$$\begin{bmatrix} a & ab \\ c & cd \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Let $f(x) = pg(x) + qg(x)h(x)$. Then $f \in \mathcal{A}$. We
compute

$$f(p_1) = pa + qab = c_1 \text{ and}$$

$$f(p_2) = pc + qcd = c_2.$$



Lemma

Let \mathcal{A} be a func. alg. Then $\bar{\mathcal{A}} \in C^0(M, \mathbb{R})$ and is a func. alg.

Pf

C^0 is closed, so $\mathcal{A} \subset C^0 \Rightarrow \bar{\mathcal{A}} \subset C^0$.

Let $f, g \in \bar{\mathcal{A}}$ and $c \in \mathbb{R}$. \exists seq's in \mathcal{A} s.t. $f_n \rightarrow f$ and $g_n \rightarrow g$. Thus,

$$\begin{aligned}(f_n + g_n) &\rightarrow f + g \Rightarrow f + g \in \bar{\mathcal{A}}, \\ cf_n &\rightarrow cf \Rightarrow cf \in \bar{\mathcal{A}}, \\ f_n \cdot g_n &\rightarrow f \cdot g \Rightarrow f \cdot g \in \bar{\mathcal{A}}.\end{aligned}$$

Thus $\bar{\mathcal{A}}$ is a func. alg. □

Lemma

$f \in \bar{\mathcal{A}} \Rightarrow |f| \in \bar{\mathcal{A}}$.

Note

$f \in \mathcal{A} \not\Rightarrow |f| \in \mathcal{A}$. Let \mathcal{A} = all polynomials. $|x^2 - 1| \notin \mathcal{A}$.

Pf

Let $f \in \bar{\mathcal{A}}$ and $M = \|f\|$. \nearrow Let $\epsilon > 0$. We apply the Weierstrass Approx. Thm to $|x| \in C^0([-M, M], \mathbb{R})$ to get a poly $p(x)$ s.t.

$$\sup \{ |p(x) - |x|| : x \in [-M, M] \} < \frac{\epsilon}{2}.$$

It follows that the constant term of the poly $p(x)$ has $|p(0)| < \epsilon/2$. Let $g(x) = p(x) - p(0)$.


Then $\sup \{ |g(x) - |x|| : x \in [-M, M] \} < \epsilon$. Let a_1, \dots, a_n be the coeff's. of $g(x)$:

$$g(x) = a_1 x + a_2 x^2 + \dots + a_n x^n.$$

Now define $g(x) = g \circ f(x) = a_1 f(x) + a_2 f^2(x) + \dots + a_n f^n(x)$.

By the previous lemma $g \in \bar{A}$. Write $y = f(x)$. Then $y \in [-M, M]$, $\forall x \in M$. Thus

$$|g(x) - |f(x)|| = |g(y) - |y|| < \epsilon.$$

Since this can be done for any $\epsilon > 0$ and \bar{A} is closed, we have $|f| \in \bar{A}$. 

Lemma

Let $f, g \in \bar{A}$. Then $\max(f, g)$ and $\min(f, g)$ are in \bar{A} .

Pf

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2} \in \bar{A}$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2} \in \bar{A} \quad \blacksquare$$

Note

By induction, this extends to \max and \min of f_1, f_2, \dots, f_n .

Pf of the S-W Thm

Let M be compact, $\mathcal{A} \subset C^0(M, \mathbb{R})$ a func. alg.
Let $F \in C^0(M, \mathbb{R})$ and let $\varepsilon > 0$. We
claim $\exists G \in \bar{\mathcal{A}}$ s.t.

$$F(x) - \varepsilon < G(x) < F(x) + \varepsilon, \quad x \in M.$$

Since this will hold $\forall \varepsilon > 0$, we get $F \in \bar{\bar{\mathcal{A}}} = \bar{\mathcal{A}}$.
Hence, $\bar{\mathcal{A}} = C^0(M, \mathbb{R})$. It will take a bit
work to get there, since, after all, we have
to use all those lemmas. Hang on to your seats!

Pick any two distinct pts $p, q \in M$. $\exists H \in \mathcal{A}$
s.t. $H(p) = F(p)$ and $H(q) = F(q)$. (First lemma)

At $x=q$ we have $\varepsilon + H(x) - F(x) = \varepsilon > 0$. By continuity
 \exists nbhd U_q of q s.t. for $x \in U_q$, $\varepsilon + H(x) - F(x) > 0$,
or equivalently $F(x) - \varepsilon < H(x)$.

Let q vary over M , and for each select an $H_q \in \mathcal{A}$
and a nbhd U_q as above: $\forall x \in U_q$, $F(x) - \varepsilon < H_q(x)$.
(p is held fixed and nothing goes "wrong" if $q=p$.)
The collection $\{U_q\}_{q \in M}$ is an open cover of M ,
and M is compact, so we can pass to a finite
subcover, say $\{U_{q_1}, U_{q_2}, \dots, U_{q_n}\}$.

Now define $G_p(x) = \max \{H_{\theta_1}(x), H_{\theta_2}(x), \dots, H_{\theta_n}(x)\}$.

Then $G_p \in \bar{A}$, $G_p(p) = F(p)$ and $F(x) - \varepsilon < G_p(x), \forall x \in M$.

At $x=p$, $\varepsilon + F(x) - G_p(x) = \varepsilon > 0$. By continuity
 \exists nbhd V_p of p s.t. $\forall x \in V_p, \varepsilon + F(x) - G_p(x) > 0$.

Thus

$$G_p(x) < F(x) + \varepsilon, \forall x \in V_p.$$

Let p vary over M . For each there is a G_p and V_p as above. The collection $\{V_p\}_{p \in M}$ is an open cover. Let $\{V_{p_1}, V_{p_2}, \dots, V_{p_m}\}$ be a finite subcover. For $i=1, 2, \dots, m$ we have

$$G_{p_i}(x) < F(x) + \varepsilon, \forall x \in V_{p_i} \text{ and}$$

$$F(x) - \varepsilon < G_{p_i}(x), \forall x \in M.$$

Let $G(x) = \min \{G_{p_1}(x), G_{p_2}(x), \dots, G_{p_m}(x)\}$.

Then $G \in \bar{A}$ and

$$F(x) - \varepsilon < G(x) < F(x) + \varepsilon, \forall x \in M.$$

Done!

Application

Fourier Series

The set of functions from $\mathbb{R} \rightarrow \mathbb{R}$ that are periodic with the same period, say 2π , can be regarded as functions from $S^1 \rightarrow \mathbb{R}$. Consider $C^0(S^1, \mathbb{R})$.

Let $\mathcal{T} \subset C^0(S^1, \mathbb{R})$ be the vector space spanned by

$$\{1, \cos(x), \cos(2x), \cos(3x), \dots, \sin(x), \sin(2x), \dots\}.$$

With the aid of some trig identities you can show \mathcal{T} is closed under multiplication. For example

$$\sin(mx) \sin(nx) = \frac{1}{2} (\cos(m-n)x - \cos(m+n)x).$$

Therefore \mathcal{T} is a func. alg. Since S^1 is compact the Stone-Weierstrass Thm says*

$$\mathcal{T} = C^0(S^1, \mathbb{R}).$$

This can be extended to function spaces that are periodic in n variables. Now the domain is the n -dimensional torus,

$$T^n = S^1 \times S^1 \times \dots \times S^1 \text{ (n times),}$$

which is compact.

* Also need nonvanishing and separates points.
(next page.)

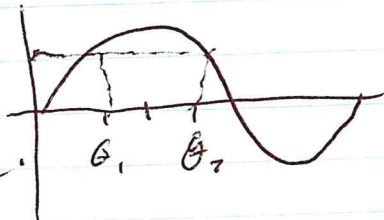
Since $l \in \mathcal{J}$ it is nonvanishing.

Let $\theta_1, \theta_2 \in S^1$, $\theta_1 \neq \theta_2$. Then $\sin \theta_1 \neq \sin \theta_2$

unless they are equal distance from $\frac{\pi}{2}$ or $3\frac{\pi}{2}$

If this is the case

use $\cos \theta$, $\cos(\theta_1) \neq \cos(\theta_2)$.



Note: We are dealing with uniform convergence. Fourier series are often used when only point-wise convergence is needed. For example, the Fourier series for $f(x)$ converge to discontinuous functions like a square wave.