Ch4

Section 6: Analytic Functions

Let $f:(a,b)\to\mathbb{R}$. Recall f is analytic at $x_0\in(a,b)$ if $\exists$ a power series $\sum c_k(x-x_0)^k$ and $\delta>0$ s.t.

$$|x-x_0|<\delta \Rightarrow \sum_{k=0}^{\infty} c_k(x-x_0)^k = f(x).$$

We have shown that ~~if~~ a function is analytic at x_0 then it is smooth at x_0, $C^\omega \subset C^\infty$. (Thm 13, Ch 4, Sec 2, pg 222). We also know that not every C^∞ function is analytic (Exercise 17, Ch 3, pg 200).

Q: We want to understand under what conditions will a smooth function by analytic.

Recall that for a power series the radius of convergence is

$$R = \left(\limsup_{k\to\infty} \sqrt[k]{|c_k|}\right)^{-1} \quad \text{(Thm 44, Ch 3, Sec 3, pg 197).}$$

Also recall that for a C^∞ func. the Taylor series centered at x_0 is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Then $R = \left(\limsup_{k\to\infty} \sqrt[k]{\frac{|f^{(k)}(x_0)|}{k!}}\right)^{-1}$, but does the series converge to f on $[-R,R]$? Uniformly? Not always as we have seen. When does it?

Let $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$.

Def Let $\sigma>0$ be s.t. $[x_0-\sigma, x_0+\sigma]\subset(a,b)$. Let $M_k=\max|f^{(k)}(x)|$ over $[x_0-\sigma, x_0+\sigma]$. Then the derivative growth rate of f over $[x_0-\sigma, x_0+\sigma]$ is defined to be

$$\alpha=\limsup_{k\to\infty}\sqrt[k]{\frac{M_k}{k!}}.$$

Clearly $\frac{1}{\alpha}\le R$.

Thm (26) Let f, σ and α be as in the above def. Let $x_0\in(a,b)$ and assume all derivatives exist on $[x_0-\sigma, x_0+\sigma]$. If $\alpha\sigma<1$ then the Taylor series of f converges uniformly to f on $[x_0-\sigma, x+\sigma]$.

Pf Let $\delta>0$ be s.t. $(\alpha+\delta)\sigma<1$. Recall the Taylor's remainder formula: $\exists\,\theta\in(x_0,x)$, or (x,x_0), s.t.

$$f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k=\frac{f^{(n)}(\theta)}{n!}(x-x_0)^n.$$

Thus $$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right|\le\frac{M_n}{n!}\sigma^n=\left(\left(\frac{M_n}{n!}\right)^{\frac{1}{n}}\sigma\right)^n$$

For large enough n the last term is $\le((\alpha+\delta)\sigma)^n$. Now $\forall\,\varepsilon>0$, $\exists N$ s.t. $n\ge N\Rightarrow((\alpha+\delta)\sigma)^n<\varepsilon$. Thus,

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\rightrightarrows f(x)\quad\text{on } [x_0-\sigma, x_0+\sigma].$$

$\boxtimes$

The next theorem uses two limits that are related to Stirling's formula, although the proofs given here are direct.

Stirling's formula

$$n! \approx \frac{n^n\sqrt{2\pi n}}{e^n} \quad \text{or}$$

$$\lim_{n\to\infty} \frac{n!\, e^n}{n^n\sqrt{2\pi n}} = 1.$$

See the paper by Keith Conrad in the links on the webpage under "Handouts"!

Limit I $\displaystyle\lim_{k\to\infty}\sqrt[k]{\frac{k^k}{k!}} = e.$

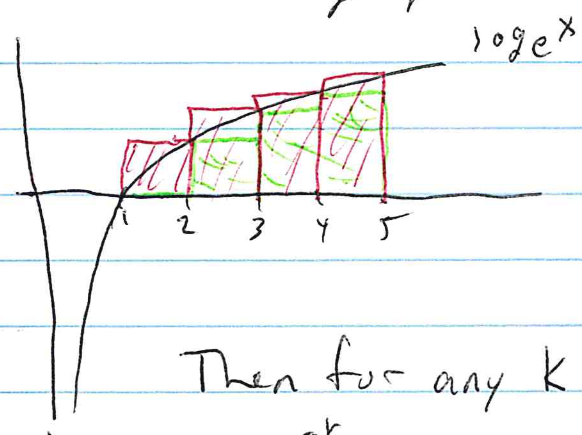
Pf We will show $\log_e\left(\frac{k}{\sqrt[k]{k!}}\right) \to 1.$

We compute

$$\log\left(\frac{k}{\sqrt[k]{k!}}\right) = \log(k) - \frac{1}{k}\log(k!) =$$

$$\log(k) - \frac{1}{k}\Big(\log(k) + \log(k-1) + \log(k-2) + \cdots + \log 3 + \log 2 + \underbrace{\log 1}_{=0}\Big).$$

Now consider the graphs below.



Then for any k we have

$$\sum_{n=1}^{k-1}\log(n) < \int_1^k \log(x)\,dx < \sum_{n=1}^{k}\log(n). \quad \star$$

This may not seem helpful at first since all three diverge to $+\infty$, but ...

Working with the right ineq in ⋆ gives,

$$\log k - \frac{1}{k}(\log k!) < \log k - \frac{1}{k}\int_1^k \log(x)\,dx$$

$$= \log k - \frac{1}{k}\Big[x\log x - x\Big]_1^k = \log k - \frac{1}{k}(k\log k - k + 1)$$

$$= 1 - \frac{1}{k}.$$

Therefore, $\lim_{k\to\infty} \log\left(\frac{k}{\sqrt[k]{k!}}\right) \leq 1.$

Now we work will the other ineq in ⋆, but we write it as

$$\sum_{n=1}^{k} \log(n) < \int_1^{k+1} \log(x)\,dx.$$

Whence,

$$\log k - \frac{1}{k}\log(k!) > \log k - \frac{1}{k}\int_1^{k+1} \log(x)\,dx =$$

$$\log k - \frac{1}{k}\Big[x\log x - x\Big]_1^{k+1} = \log k - \frac{1}{k}\Big[(k+1)\log(k+1) - (k+1) + 1\Big]$$

$$= \log k - \frac{k+1}{k}\log(k+1) + \frac{k+1}{k} - \frac{1}{k} = \log k - \log(k+1) - \frac{1}{k}\log(k+1) + 1 + \frac{1}{k} - \frac{1}{k}$$

$$= \log\left(\frac{k}{k+1}\right) - \frac{\log(k+1)}{k} + 1 \longrightarrow \log(1) - 0 + 1 = 1.$$

L'Hopital's Rule

Thus, $\lim_{k\to\infty} \log\left(\frac{k}{\sqrt[k]{k!}}\right) \geq 1.$

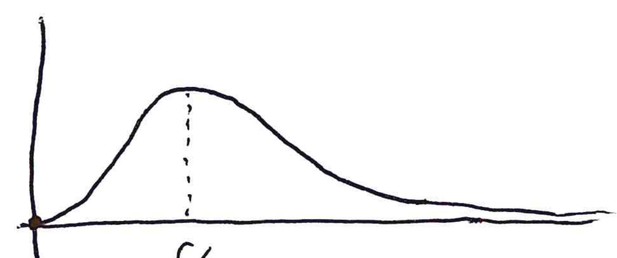
Thus, $\lim_{k\to\infty} \log\left(\sqrt[k]{\frac{k^k}{k!}}\right) = 1$. Hence $\lim_{k\to\infty} \sqrt[k]{\frac{k^k}{k!}} = e$. $\boxtimes$

Limit II For $\lambda \in (0,1)$, we have $\limsup_{r\to\infty} \sqrt{\sum_{k=r}^{\infty} \binom{k}{r} \lambda^k}$ converges.

pf Let $\mu = -\ln\lambda$. Then $\lambda = e^{-\mu}$ with $\mu \in (0,\infty)$.

$$\sum_{k=r}^{\infty} \binom{k}{r} \lambda^k = \sum_{k=r}^{\infty} \frac{k(k-1)(k-2)\cdots(k-r+1)}{r!} e^{-k\mu} \leq \frac{1}{r!} \sum_{k=r}^{\infty} k^r e^{-k\mu}.$$

Let $f(x) = x^r e^{-\mu x}$. The graph of $y = f(x)$ is below.



In the case that $\mu \geq 1$, $r/\mu \leq r$. Thus we can choose a staircase function based on $k^r e^{-k\mu}$ that is always below $f(x)$. Thus, if $\int_r^\infty f(x)\,dx$ converges, then $\sum_{k=r}^{\infty} k^r e^{-k\mu}$ will too.

If $\mu \in (0,1)$, then $\frac{r}{\mu} > r$. But, once $x > \frac{r}{\mu}$ we can have a staircase function based on our sum that is below $f(x)$. This is enough so that ~~con~~ convergence of the integral forces convergence of the sum.

This is what the author of the textbook means when he writes $\sum_{k=r}^{\infty} k^r e^{-k\mu} \sim \int_r^\infty x^r e^{-\mu x}\,dx$.

Now we study $\int_r^\infty x^r e^{-\mu x}\,dx$. You can use induction on r to show that it converges and

$$\star = \int_r^\infty x^r e^{-\mu x}\,dx = e^{-\mu r}\left(\frac{r^r}{\mu} + \frac{r^r}{\mu^2} + \frac{(r-1)r^{r-1}}{\mu^3} + \frac{(r-1)(r-2)r^{r-2}}{\mu^4} + \cdots + \frac{r!}{\mu^{r+1}}\right).$$

Before taking the limit of $\frac{1}{r!}\int_r^\infty x^r e^{-\mu x}\,dx$ we make some estimates. Each numerator is $\leq r^r$. Thus,

$$\star \leq e^{-\mu r} r^r \left(\frac{1}{\mu} + \frac{1}{\mu^2} + \frac{1}{\mu^3} + \cdots + \frac{1}{\mu^{r+1}}\right).$$

If $\mu \in (0,1]$ then $\frac{1}{\mu^{r+1}}$ is larger than the other terms in the sum. Thus,

$$\star \leq e^{-\mu r} r^r (r+1)\left(\frac{1}{\mu^{r+1}}\right).$$

If $\mu > 1$, then each term is < 1. Thus

$$\star \leq e^{-\mu r} r^r (r+1).$$

Let $\alpha = \min\{1, \mu\}$. Then, in either case,

$$\star \leq e^{-\mu r} r^r (r+1)\left(\frac{1}{\alpha}\right)^{r+1}.$$

Putting this all back together, we have

$$\sum_{k=r}^\infty \binom{k}{r}\lambda^k \leq \frac{1}{r!} e^{-\mu r}(r+1) r^r \left(\frac{1}{\alpha}\right)^{r+1}.$$

Take the r^{th} root of both sides.

$$\sqrt[r]{\sum_{k=r}^{\infty} \binom{k}{r} \lambda^k} \leq \sqrt[r]{\frac{r^r}{r!}}\ e^{-\mu} (r+1)^{\frac{1}{r}} \left(\frac{1}{\alpha}\right)^{\frac{r+1}{r}} \quad (\#)$$

You can check that $(r+1)^{\frac{1}{r}} \to 1$ and $\left(\frac{1}{\alpha}\right)^{\frac{r+1}{r}} \to \frac{1}{\alpha}$ as $r \to \infty$. By Limit I $\sqrt[r]{\frac{r^r}{r!}} \to e$. Thus,

$$\lim_{r\to\infty} (\text{RHS of } \#) = e^{1-\mu}/\alpha.$$

Thus, the sequence is bounded. Thus, by exercise 45 on pg 52, the lim sup exists.

Thm (27) If $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ has radius of convergence R and $0 < \sigma < R$, then $f(x)$ has bounded derivative growth rate, α, on $[x_0 - \sigma, x_0 + \sigma]$.

pf $\exists N \in \mathbb{N}$ s.t. $k \geq N \Rightarrow |c_k|^{\frac{1}{k}} \leq R$. Pick λ s.t. $\frac{\sigma}{R} < \lambda < 1$. Then $|c_k|^{\frac{1}{k}} \sigma < \lambda$, so $|c_k \sigma^k| < \lambda^k$.

Differentiate the given power series term-by-term to get

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, c_k (x - x_0)^{k-n}.$$

Thus,

$$|f^{(n)}(x)| \leq \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, |c_k|\, |x - x_0|^{k-n} = (*).$$

We write $k(k-1)(k-2)\cdots(k-n+1) = \frac{k!}{(k-n)!} = n!\left(\frac{k!}{n!(k-n)!}\right) = n!\binom{k}{n}$.

Recall we are assuming $|x - x_0| \leq \sigma$. Thus

$$(*) \leq n! \sum_{k=n}^{\infty} \binom{k}{n} |c_k| \sigma^{k-n} = \frac{n!}{\sigma^n} \sum_{k=n}^{\infty} \binom{k}{n} |c_k \sigma^k|$$

$$\leq \frac{n!}{\sigma^n} \sum_{k=n}^{\infty} \binom{k}{n} \lambda^k$$

for $n \geq N$.

Thus, $M_n = \sup_{x \in [x_0-\sigma,\, x_0+\sigma]} |f^{(n)}(x)| \le \frac{n!}{\sigma^n} \sum_{k=n}^{\infty} \binom{k}{n} \lambda^k$.

Now,
$$\sqrt[n]{\frac{M_n}{n!}} \le \frac{1}{\sigma} \sqrt[n]{\sum_{k=n}^{\infty} \binom{k}{n} \lambda^{1/k}}$$

By Limit II we get

$$\limsup_{n\to\infty} \frac{1}{\sigma} \sqrt[n]{\sum_{k=n}^{\infty} \binom{k}{n} \lambda^{1/k}} < \infty,$$

Thus,
$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{M_n}{n!}} < \infty,$$

and so f has bdd derivative growth rate on $[x_0-\sigma, x_0+\sigma]$ as claimed.

Thm(28) (Analyticity Thm) A smooth (C^∞) function is analytic iff it has locally bdd der. growth rate.

Pf In essence Thm 26 gives one direction, locally bdd der. growth rate $\Rightarrow$ analytic, and Thm 27 gives the only. See textbook for details.

~~This~~ $(\Rightarrow)$ can be restated as

Cor 29: A smooth func. is analytic if its derivatives are uniformly bdd. See textbook for proof.

Thm 30 (Taylor's Thm) If $f(x) = \sum c_k x^k$ and the power series has radius of conv. R, then f is analytic on $(-R, R)$.

pf See textbook, page 251.

A Brief Detour into Complex Functions and Series

Def Let $\mathbb{C}$ be the complex plane (same metric as $\mathbb{R}^2$, but $\mathbb{C}$ is a field whereas $\mathbb{R}^2$ is only a vector space). Let $U \subset \mathbb{C}$ and let $f: U \to \mathbb{C}$. Assume $z_0 \in \text{int}(U)$. $\exists\ \epsilon > 0$ s.t. $B(z_0, \epsilon) \subset \text{int}(U)$. Then the derivative of $f(z)$ at z_0 is

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h},$$

when the limit exists. Note, h is complex. $h \to 0$ means $\text{Re}(h), \text{Im}(h) \to 0$, and is independent of the path to $0 + 0i$.

The derivative formulas you are used to still hold. Differentiability still implies continuity. But this is not simply having $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ existing. Instead we have...

Thm Let $w = f(z)$ where $f: \mathbb{C} \to \mathbb{C}$. Write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Assume f is diff'able at $z_0 = x_0 + iy_0$. Then the partial derivatives, $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$, $v_y(x_0,y_0)$ exist and

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{Cauchy-Riemann equations.}$$

And, these are sufficient.

Pf Short and easy. See any Complex Analysis textbook.

And now something wonderful happens...

Thm If $f(z)$ is diff'able in some ε-nbhd of z_0 then

$f(z)$ is infinitely diff'able and the Taylor series converges to $f(z)$.

That is $\boxed{\text{differentiability} \Rightarrow \text{analytic !!}}$

The proof, covered in 455, uses line integrals. It also turns out that any diff'able func. has path independence!

$$\oint_C f(z)\,dz = 0.$$

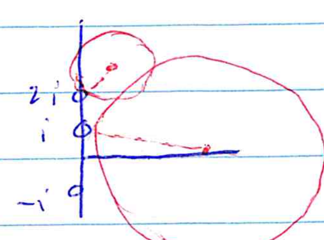
(f must be diff'able inside and on the curve C)

Also, there is a handy method to find the radius of convergence.

If f is analytic everywhere, $R = \infty$.
If not but z_0 is a point where f is analytic then R for the Taylor series centered at z_0 is the min. distance from z_0 to a point where f is not analytic.

Ex Let $f(z) = \sin^3(z)\, e^z / (z^2+1)(z-2i)$.

If $z_0 = 0$, $R = 1$.
If $z_0 = 7$, $R = \sqrt{49+1}$
If $z_0 = 5i$, $R = 3$.



This helps explain something that puzzles students in Calc II. Let

$$f(x) = \frac{1}{1-x^2} \quad \text{and} \quad g(x) = \frac{1}{1+x^2}.$$

Take their Taylor series centered at $x=0$. They can see why $R_f = 1$, but why should R_g also be 1? The answer is in the complex plane!

If we use $x_0 = 2$, $R_f = 1$ but $R_g = \sqrt{5}$.