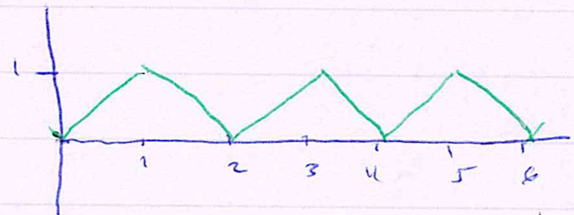
Ch 4

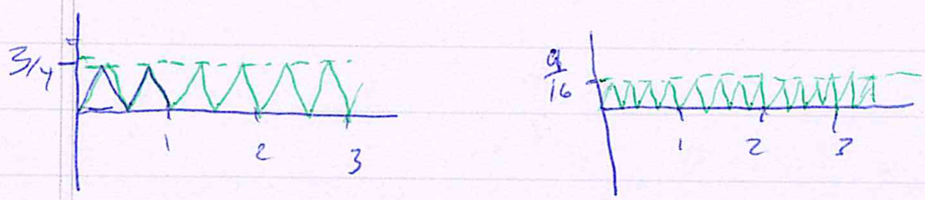
Section 7: Nowhere Differentiable Functions are everywhere!

Thm $\exists f:\mathbb{R}\to\mathbb{R}$ that is continuous, but nowhere differentiable.

Pf Let $\sigma_0(x) = \begin{cases} x-2n & \text{if } 2n \le x \le 2n+1 \\ (2n+2)-x & \text{if } 2n+1 \le x \le 2n+2. \end{cases}$



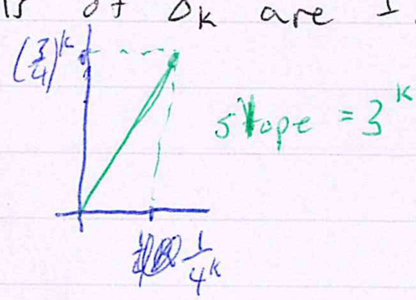
It is continuous. $\sigma_k(x) = \left(\frac{3}{4}\right)^k \sigma_0(4^k x)$. The period of σ_k is $\pi_k = 2/4^k$.



According to the Weierstrass M-test (pg 217), since $\|\sigma_k\| = \left(\frac{3}{4}\right)^k$ and $\sum\left(\frac{3}{4}\right)^k$ converges the sum $\sum_{k=0}^{\infty}\sigma_k(x)$ converges uniformly to a continuous function. Let

$$\sum_{k=0}^{\infty}\sigma_k(x) \rightrightarrows f(x).$$

We claim $f'(x)$ does not exist $\forall x\in\mathbb{R}$. The proof basically rests on the following: the slopes of the linear segments of σ_k are $\pm 3^k$.



Let $x \in \mathbb{R}$ and $\delta > 0$. Let $\delta_n = \frac{1}{2 \cdot 4^n}$. $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \delta_n < \delta$. We define a slope function

$$S(\alpha) = \frac{f(x+\alpha) - f(x)}{\alpha},$$

for $\alpha \in (-\delta, \delta)$. We will show that for $n \geq N$ either $|S(\delta_n)|$ or $|S(-\delta_n)|$ is $\geq \frac{1}{2}(3^n + 1)$. Thus the limit $\lim_{\alpha \to 0} S(\alpha)$ does not exist.

Set $\alpha = \pm \delta_n$. Then $|\alpha| = \delta_n = \frac{1}{2 \cdot 4^n} = 4^{k-(n+1)}\left(\frac{2}{4^k}\right) = 4^{k-(n+1)} \pi_k$.

Thus, once $k > n$ we have $\sigma_k(x \pm \delta_n) = \sigma_k(x)$. Thus

$$S(\alpha) = \sum_{k=0}^{\infty} \frac{\sigma_k(x+\alpha) - \sigma_k(x)}{\alpha} = \sum_{k=0}^{n} \frac{\sigma_k(x+\alpha) - \sigma_k(x)}{\alpha}$$

Consider the last nonzero term, $k = n$,

$$\frac{\sigma_n(x \pm \delta_n) - \sigma_n(x)}{\pm \delta_n}.$$ On either $[x - \delta_n, x]$ or $[x, x+\delta_n]$

has σ_n monotone. If the first does we use $\alpha = -\delta_n$, if not use $\alpha = \delta_n$. Thus, we have

$$\left| \frac{\sigma_n(x+\alpha) - \sigma_n(x)}{\alpha} \right| = 3^n.$$

For $0 \le k < n$ we have $\left| \dfrac{\sigma_k(x \pm \delta_n) - \sigma_k(x)}{\pm \delta_n} \right| \le 3^k$.

Thus,

$$|S(\alpha)| \geqslant 3^n - \left(3^{n-1} + 3^{n-2} + 3^{n-3} + \cdots + 3 + 1\right)$$

$$= 3^n - \left(\frac{3^n - 1}{3 - 1}\right) = \frac{1}{2}\left(3^n + 1\right).$$

Therefore $f'(x)$ does not exist. $\blacksquare$

This result, by itself, is not all that ~~surprising~~ surprizing. What is, is that "most" cont. functions are nowhere differentiable! We need some definitions to make the idea "most" precise. These will lead to Baire's Thm.

Let X be a topological space. For each $n \in \mathbb{N}$ let $G_n \subset X$ be open and dense. Let $G = \bigcap G_n$. Sets constructed in this way are sometimes called **residual sets**. We are interested in what conditions on X will gavrentee that residual sets are themselves dense. Spaces with this property are called **Baire Spaces**.

Suppose each point of X either has a certain property or does not. This property is said to be **generic** if every point in some residual set has it.

Ex Let $\{q_1, q_2, \dots\}$ be an ordering of the rational numbers. Let $G_n = \mathbb{R} - q_n$. Then each G_n is open and dense. Their intersection, $\bigcap G_n$, is the set of irrational numbers. Thus being irrational is generic. In this case $\overline{\bigcap G_n} = \mathbb{R}$.

See textbook for several other examples.

We will show that being nowhere differentiable is a generic property of $C^0([a,b], \mathbb{R})$.

Thm (Baire's Thm) Let M be a complete metric space. Let $\{G_k\}_{k=1}^{\infty}$ be a countable collection of open dense subsets of M. Then $G = \bigcap_{k=1}^{\infty} G_k$ is dense in M.

Pf (From Royden's Real Analysis, pg 158)

Let U be an arbitrary open subset of M. We will show that $U \cap G$ is nonempte and hence $\overline{G} = M$. This will be done by constructing a sequence $\{x_n\}$ whose limit must be in $U \cap G$.

Let $x_1 \in G_1 \cap U$, which is nonempty since G_1 is dense. Let B_1 be ~~the~~ an open ball $B(x_1, r_1)$ with $r_1 > 0$ small enough that $B_1 \subset G_1 \cap U$.

Let $x_2 \in G_2 \cap B_1$. Let B_2 be an open ball $B(x_2, r_2)$ s.t. $B_2 \subset G_2$, $r_2 < \frac{r_1}{2}$ and $r_2 < r_1 - d(x_1, x_2)$. This insures that $\overline{B}_2 \subset B_1$.

We continue inductively, forming sequences

$$\{x_n\} \text{ and } \{B_n\} \text{ s.t.}$$

$$B_n \subset G_n, \quad \overline{B}_n \subset B_{n-1} \quad \text{and} \quad r_n \to 0.$$

We claim $\{x_n\}$ is Cauchy. To see this just note that $m, n \geq N \Rightarrow x_n, x_m \in B_N \Rightarrow d(x_n, x_m) < 2r_N$. Since $r_N \to 0$, $\{x_n\}$ is Cauchy.

Since M is complete, $\exists x \in M$ s.t. $x_n \to x$.

Now we show that $x \in G$. For $n > N$, $x_n \in B_{N+1}$. Thus,

$$x \in \overline{B_{N+1}} \subset B_N \subset G_N.$$

This holds for all $N \in \mathbb{N}$. Thus $x \in G_N \ \forall N \in \mathbb{N}$, meaning $x \in G$.

$\square$

Rmk Baire's Thm hold if the condition "complete metric space" is replaced with "compact Hausdroff space" (which may not even have a metric.) See Munkres' Topology, Section 48.

Rmk The definition of a Baire space is equivalent to the following: Given any countable collection $\{A_n\}$ of closed sets in M with empty interiors, their union, $\bigcup A_n$, also has empty interior. See Munkres', Sec 48.

Other terminology

Residual subsets are also called thick subsets. The complement of a residual set is called a meager set or a thin set. [Pugh's textbook, pg 256.]

A subset of a topological space is said to be of the first category if it was contained in the union of a countable collection of closed sets having empty interiors. Otherwise it is of the second category. [Munkres Topology pg 295.]

In this terminology Baire's Thm is called Baire's Category Thm: In a Baire space no nonempty open subset is of the first category.

Thm In $C^0([0,1],\mathbb{R})$ the set of nowhere differentiable functions is residual and dense.

Pf* Let $U_n = \{ f \in C^0 \mid \forall t \in [0,1],\ \exists s \in [0,1] - \{t\},$

$$\text{s.t. } \left| \frac{f(t)-f(s)}{t-s} \right| > n \}.$$

If $f(x) = 2n$, then $f \in U_n$. Hence $U_n \neq \emptyset$.

The proof is done in the following steps.

Claim I If $f \in \bigcap U_n$ then f is nowhere differentiable.

Claim II Each U_n is dense.

Claim III Each U_n is open.

The result follows by Baire's Thm.

* We are using the proof from Topology and Geometry by Glen Bredon, Corollary 17.6, pages 60-61.

Claim I Let $f \in \bigcap U_n$. Suppose $f'(t)$ exists, for some $t \in [0,1]$. Consider $\left|\frac{f(t)-f(s)}{t-s}\right|$ as a function of s. $\exists$ $\delta > 0$ s.t.

$$s \in \big((t-\delta, t) \cup (t, t+\delta)\big) \cap [0,1] \implies \left|\frac{f(t)-f(s)}{t-s}\right| < f'(t)+1.$$

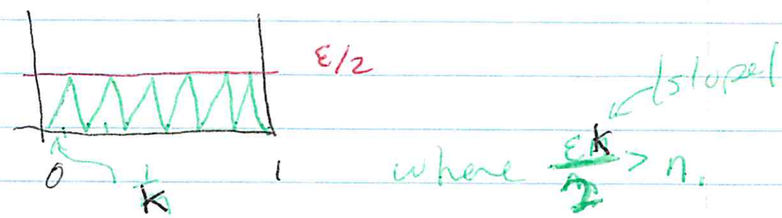
On the compact set $[0,1] \setminus (t-\delta, t+\delta)$

$\left|\frac{f(t)-f(s)}{t-s}\right|$ is cont. and hence bdd, by say M.

Let $n > \max\{M, f'(t)+1\}$. Then $f \notin U_n$.

This is a contradiction. Hence $f'(t)$ does not exist for any $t \in [0,1]$.

Claim II Each U_n is dense. Let $f \in C^0$ and $\varepsilon > 0$. We need to construct a $g \in U_n$ s.t. $\|f - g\| < \varepsilon$. If f was the zero function we know how to do this. Let $g(x)$ be given by



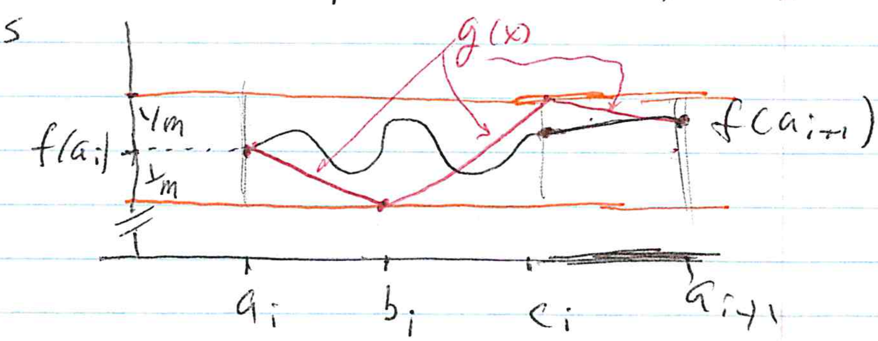
In general pick m s.t. $2/m < \varepsilon$. By uniform continuity of f $\exists$ k s.t.

$$|x - y| \leq \frac{1}{k} \Rightarrow |f(x) - f(y)| \leq \frac{1}{m}.$$

Choose k even larger, if need be, so that $k > mn$.

Let $a_i = \frac{i}{k}$, $i = 0, \dots k$. This gives a partition of $[0,1]$.
Let $b_i = a_i + \frac{1}{3k}$ and $c_i = a_i + \frac{2}{3k}$, $i = 0, \dots, k-1$.

Define $g(x)$ on each partition $[a_i, a_{i+1}]$ like this



This defined $g : [0,1] \to \mathbb{R}$ that is continuous and

$$\|f - g\| \leq \frac{2}{m} < \varepsilon.$$

We examine the three segments that make up g on $[a_i, a_{i+1}]$.

Suppose $t \in [a_i, b_i]$. If we choose $s \neq t$ in $[a_i, b_i]$

$$\left| \frac{f(t)-f(s)}{t-s} \right| = |\text{slope}| = \left| \frac{-1/m}{1/3k} \right| = \frac{3k}{m} > \frac{3nm}{m} = 3n > n.$$

Suppose $t \in [b_i, c_i]$. If we choose $s \neq t$ in $[b_i, c_i]$

$$\left| \frac{f(t)-f(s)}{t-s} \right| = |\text{slope}| = \left| \frac{2/m}{1/3k} \right| = \frac{6k}{m} > \frac{6mn}{m} = 6n > n.$$

For $t \in [c_i, a_{i+1}]$ we have to consider two cases.

(i) $f(t) \geq f(a_i)$, let $s = b_i$. Then

$$\left| \frac{f(b_i)-f(t)}{b_i - x} \right| \geq \frac{|f(b_i)-f(x)|}{2/3k} \geq \frac{1/m}{2/3k}$$

$$= \frac{3k}{2m} < \frac{3mn}{2m} = \frac{3}{2}n > n.$$

(ii) $f(t) < f(a_i)$, Then $f(a_{i+1}) < f(a_i)$.

Thus for any $s \neq t$ in $[c_i, a_{i+1}]$

$$\left| \frac{f(t)-f(s)}{t-s} \right| = |\text{slope}| \geq \left| \frac{-1/m}{1/3k} \right| = \frac{3k}{m} > n.$$

Thus $g \in U_n$.

Claim III Each U_n is open. Let $f \in U_n$. We need to find a $\eta > 0$ s.t. $\|f - g\| < \eta \Rightarrow g \in U_n$.

For each $t \in [0,1]$, $\exists\ s$ s.t. $\left| \frac{f(t) - f(s)}{t - s} \right| > n$.

Since s depends on t, we may write $s(t)$.

$\exists\ \varepsilon(t)$ s.t. $\left| \frac{f(t) - f(s)}{t - s} \right| > n + \varepsilon$. By continuity,

$\exists$ nbhd V_t of t s.t. $\forall\ t' \in V_t$ we have

$$\left| \frac{f(t') - f(s)}{t' - s} \right| > n + \varepsilon.$$

We can shrink V_t so that $s(t) \notin \overline{V_t}$.

Do this for each $t \in [0,1]$. Then $\{V_t\}_{t \in [0,1]}$ is an open covering of $[0,1]$. $\exists$ a finite subcover $\{V_{t_1}, V_{t_2}, \dots, V_{t_k}\}$. Let

$$\varepsilon = \min \{\varepsilon(t_i)\}_{i=1}^{k},$$

$$\delta = \min_{i=1\dots k} \operatorname{dist}(s(t_i), \overline{V}_{t_i}) \neq 0, \text{ and}$$

$$\eta = \varepsilon\delta/2.$$

For any $t \in [0,1]$, $t \in V_{t_i}$ for some i. Let $s = s(t_i)$.

Then

$$n+\varepsilon < \left|\frac{f(t)-f(s)}{t-s}\right| \le \left|\frac{f(t)-g(t)}{t-s}\right| + \left|\frac{g(t)-g(s)}{t-s}\right| + \left|\frac{g(s)-f(s)}{t-s}\right|.$$

We have $|f(t)-g(t)|$ and $|g(s)-f(s)|$ are $< \varepsilon\delta/2$, and that $|t-s| \geq \delta$. Thus

$$n+\varepsilon < \frac{\varepsilon}{2} + \left|\frac{g(t)-g(s)}{t-s}\right| + \frac{\varepsilon}{2}.$$ Thus

$$\left|\frac{g(t)-g(s)}{t-s}\right| > n+\varepsilon-\varepsilon = n.$$

Hence, $\boxed{g \in U_n}$.