An Intro to Ergodic Theory

Def Let $(X, \mathcal{M}, m)$ be a set X with a σ-alg. $\mathcal{M}$ and a measure m. If $m(X)$ we call this triple a probability space.

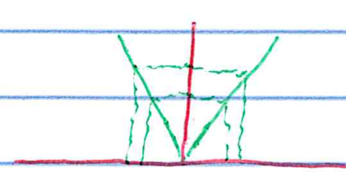
Def Let $X = G$ be a compact topological group. Let $\mathcal{B}(G)$ be the σ-alg. of Borel subsets of G. Then the measure m, under certain conditions, is called a Haar measure.

The most common examples are S^1 and n-dimensional tori $T^n = S^1 \times \cdots \times S^1$, where the group operations are rotations.

Def Let $T: X \to X$. If $m(T^{-1}(B)) = m(B)$, $\forall B \in \mathcal{M}$, we say that T is measure preserving.

Ex The identity map $T(x) = x$ is obviously meas. pres.

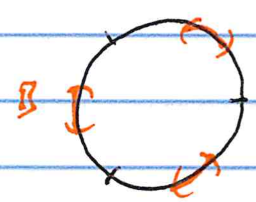
For $X = \mathbb{R}$ $T(x) = |2x|$ is meas. pres.



Def Let $(X, \mathcal{M}, m)$ be a prob. sp. A meas. pres. $T : X \to X$ is $\boxed{\text{ergodic}}$ if the only members $B \in \mathcal{M}$ s.t. $T^{-1}(B) = B$ have $m(B) = 0$ or $m(B) = 1$.

The idea is $T^{-1}(B) = B$ can only happen when B is almost all of X or almost none of X. This means any "interesting" subset is getting smeared around.

Ex Rotations of a circle are ergodic iff they are irrational (wrt to 2π).



A rotation by $1/3$ is not ergodic.

Thm The following are equivalent

(a) T is ergodic.

(b) $\forall A \in \mathcal{B}$ with $m(A) > 0$ we have $m\left(\bigcup_{n=1}^{\infty} T^{-n}(A)\right) = 1$.

(c) $\forall A, B \in \mathcal{B}$ with $m(A) > 0$, $m(B) > 0$ $\exists n > 0$ with
$$m(T^{-n}(A \cap B)) > 0.$$

Def Let $L^p(X, \mathcal{M}, m) =$ all measurable functions $f: X \to \begin{smallmatrix}\mathbb{C}\\ \mathbb{R}\end{smallmatrix}$ s.t.

$$\int_X |f|^p < \infty.$$

Then $T: X \to X$ induces a map $U_T: L^p \to L^p$ via

$$U_T(f)(x) = f(T(x)).$$

Thm Let $f \in L^1$. and $f_n = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$.

Then $f_n \xrightarrow{a.e.} f^* \in L^1$ and $f^* \circ T = f^*$ a.e. and

$$\int f^* = \int f.$$

Furthermore, if T is ergodic f^* is a constant and so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \int f, \quad \text{a.e. } x \in X.$$

Thm For almost all numbers in $[0,1)$ the frequency of 1's in the binary expansion is $\frac{1}{2}$. (We show what this means in the proof.)

Outline of Pf Let $T:[0,1)\to[0,1)$ by $T(x) = 2x \bmod 1$. It is known that T is ergodic.

Let Y denote the set of pts of $[0,1)$ that have a unique binary expansion. Since $[0,1)\setminus Y$ is countable we have $m(Y)=1$.

Let $x \in Y$ and $x = \frac{a_1}{2} + \frac{a_2}{4} + \cdots$, $a_i \in \{0,1\}$. Then

$$T(x) = \frac{a_2}{2} + \frac{a_3}{4} + \cdots$$

Let $f(x) = \chi_{[\frac{1}{2},1)}(x)$. Then

$$f(T^i(x)) = f\left(\frac{a_{i+1}}{2} + \frac{a_{i+2}}{4} + \cdots\right) = \begin{cases} 1 & \text{if } a_{i+1} = 1 \\ 0 & \text{if } a_{i+1} = 0. \end{cases}$$

For $x \in Y$, the number of 1's in the first n digits is

$$\sum_{i=0}^{n-1} f(T^i(x)). \quad \text{But,}$$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) \xrightarrow{a.e.} \int \chi_{[\frac{1}{2},1)} = \frac{1}{2}.$$

$\blacksquare$

References

An Introduction to Ergodic Theory
Peter Walters, Springer

Fundamentals of Measurable Dynamics: Ergodic Theory on Lebesgue Spaces
Daniel J. Rudolph, Clarendon Press